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MULTIPLICATIVE MAPPINGS OF ALTERNATIVE RINGS

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Abstract

Let $\mathfrak{R}$ and $\mathfrak{S}$ be arbitrary alternative rings. A mapping φ of $\mathfrak{R}$ onto $\mathfrak{S}$ is called a multiplicative isomorphism if φ is bijective and satisfies $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \mathfrak{R}$. In this paper we establish a condition on $\mathfrak{R}$ that assures that φ is additive, i.e., $\varphi(x + y) = \varphi(x) + \varphi(y)$ for all $x, y \in \mathfrak{R}$. As an application we generalize the famous Martindale's result.

1 Preliminaries

In this paper, $\mathfrak{R}$ will be a ring not necessarily associative or commutative. A ring $\mathfrak{R}$ is called *k-torsion free* if $kx = 0$ implies $x = 0$ for any $x \in \mathfrak{R}$, where $k \in \mathbb{Z}$, $k > 0$, and *prime* if $\mathfrak{I}\mathfrak{K} \neq 0$ for any two nonzero ideals $\mathfrak{I}, \mathfrak{K} \subseteq \mathfrak{R}$.

Let $\mathfrak{R}$ and $\mathfrak{S}$ be arbitrary rings and let φ denote a mapping of $\mathfrak{R}$ into $\mathfrak{S}$. The mapping φ is called a *multiplicative mapping* if $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \mathfrak{R}$, and a *multiplicative isomorphism* if in addition φ is bijective. We call φ an *additive mapping* when $\varphi(x+y) = \varphi(x) + \varphi(y)$ for all $x, y \in \mathfrak{R}$.

The study of the question of when a multiplicative isomorphism is additive has become an active research area in associative ring theory. In this case, one often tries to establish conditions on the ring which assures the additivity of every multiplicative isomorphism defined on it. The first result in this direction is due to Martindale III [4] who obtained a pioneer result in 1969, which in his condition requires that the ring possess idempotents. Recently, the Martindale's result was generalized in [3]. Their authors established a condition on the ring, in the case where it may not contain any nonzero idempotents, and they assured the additivity of every multiplicative isomorphism.

For these mappings defined on an arbitrary ring it is very difficult to say anything. However in the present papers we take up Fang Yan Lu and Jin Hai Xie's ideas [3] and we investigate the problem of when a multiplicative isomorphism is additive for the class of alternative rings.

2 Alternative rings and the multiplicative isomorphisms

A ring $\mathfrak{R}$ is said to be *alternative* if

$$(x, x, y) = 0 = (y, x, x), \text{ for all } x, y \in \mathfrak{R},$$

where $(x, y, z) = (xy)z - x(yz)$ is the *associator* of x, y and z .

One easily sees that any associative ring is alternative.

Let $\mathfrak{R}$ be an alternative ring. For each nontrivial idempotent $e_1 \in \mathfrak{R}$, i.e., $e_1^2 = e_1$ and e_1 is a nonzero element different from the multiplicative identity

of $\mathfrak{R}$ (note that $\mathfrak{R}$ need not have an identity element), let $e_2: \mathfrak{R} \rightarrow \mathfrak{R}$ and $e'_2: \mathfrak{R} \rightarrow \mathfrak{R}$ be given by $e_2a = a - e_1a$ and $e'_2a = a - ae_1$. We denote e'_2a by ae_2 (allowing us to write $1 = e_1 + e_2$). It is easy to see that $(e_ia)e_j = e_i(ae_j)$ for all $a \in \mathfrak{R}$ and $i, j = 1, 2$. Then $\mathfrak{R}$ has a Peirce decomposition (to see [2]) $\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{21} \oplus \mathfrak{R}_{22}$, where $\mathfrak{R}_{ij} = e_i\mathfrak{R}e_j$ ($i, j = 1, 2$), satisfying the following multiplicative relations:

- (i) $\mathfrak{R}_{ij}\mathfrak{R}_{jl} \subseteq \mathfrak{R}_{il}$ ($i, j, l = 1, 2$);
- (ii) $\mathfrak{R}_{ij}\mathfrak{R}_{ij} \subseteq \mathfrak{R}_{ji}$ ($i, j = 1, 2; i \neq j$);
- (iii) $\mathfrak{R}_{ij}\mathfrak{R}_{kl} = 0$, if $j \neq k$ and $(i, j) \neq (k, l)$ ($i, j, k, l = 1, 2$);
- (iv) $x_{ij}^2 = 0$ for all $x_{ij} \in \mathfrak{R}_{ij}$ ($i, j = 1, 2; i \neq j$).

Let us state our main theorem.

Theorem 2.1. *Let $\mathfrak{R}'$ be an alternative ring containing a family $\{e_\alpha | \alpha \in \Lambda\}$ of idempotents. Suppose that $\mathfrak{R}$ is a subring of $\mathfrak{R}'$ which satisfies:*

- (i) *For each $\alpha \in \Lambda$, $e_\alpha\mathfrak{R} \subseteq \mathfrak{R}$ and $\mathfrak{R}e_\alpha \subseteq \mathfrak{R}$;*
- (ii) *If $x \in \mathfrak{R}$ is such that $x\mathfrak{R} = 0$, then $x = 0$;*
- (iii) *If $x \in \mathfrak{R}$ is such that $(e_\alpha\mathfrak{R})x = 0$ (or $e_\alpha(\mathfrak{R}x) = 0$) for all $\alpha \in \Lambda$, then $x = 0$ (and hence $\mathfrak{R}x = 0$ implies $x = 0$);*
- (iv) *For each $\alpha \in \Lambda$ and $x \in \mathfrak{R}$, if $(e_\alpha x e_\alpha)(\mathfrak{R}(1 - e_\alpha)) = 0$ then $e_\alpha x e_\alpha = 0$.*

Then any multiplicative isomorphism φ of $\mathfrak{R}$ onto an arbitrary alternative ring is additive.

The following lemmas have the same hypotheses of Theorem 2.1 and we need these lemmas to the proof this theorem.

Lemma 2.2. $\varphi(0) = 0$.

Proof. Since φ is onto we can choose $x \in \mathfrak{R}$ such that $\varphi(x) = 0$. Thus $\varphi(0) = \varphi(0x) = \varphi(0)\varphi(x) = \varphi(0)0 = 0$. □

The conditions (iii) and (iv) ensure that Λ has a nontrivial idempotent. Let $e_1 \in \Lambda$ be a nontrivial idempotent.

Lemma 2.3. $\varphi(x_{ii} + x_{jk}) = \varphi(x_{ii}) + \varphi(x_{jk})$, $j \neq k$.

Proof. First, let us assume that $i = j = 1$ and $k = 2$. Since φ is onto, let z be an element of $\mathfrak{R}$ such that $\varphi(z) = \varphi(x_{11}) + \varphi(x_{12})$ and let us write $z = z_{11} + z_{12} + z_{21} + z_{22}$. For all $a_{11} \in \mathfrak{R}_{11}$ we have $\varphi(za_{11}) = \varphi(z)\varphi(a_{11}) = (\varphi(x_{11}) + \varphi(x_{12}))\varphi(a_{11}) = \varphi(x_{11}a_{11}) + \varphi(x_{12}a_{11}) = \varphi(x_{11}a_{11}) + \varphi(0) = \varphi((x_{11} + x_{12})a_{11})$. Hence $za_{11} = (x_{11} + x_{12})a_{11}$, that is,

$$(z - (x_{11} + x_{12}))a_{11} = 0. \quad (1)$$

Next, for all $a_{21} \in \mathfrak{R}_{21}$ we have $\varphi(za_{21}) = \varphi(z)\varphi(a_{21}) = (\varphi(x_{11}) + \varphi(x_{12}))\varphi(a_{21}) = \varphi(x_{11}a_{21}) + \varphi(x_{12}a_{21}) = \varphi(0) + \varphi(x_{12}a_{21}) = \varphi((x_{11} + x_{12})a_{21})$ which implies $za_{21} = (x_{11} + x_{12})a_{21}$. So

$$(z - (x_{11} + x_{12}))a_{21} = 0. \quad (2)$$

Yet, for all $a_{22} \in \mathfrak{R}_{22}$ we have $\varphi(za_{22}) = \varphi(z)\varphi(a_{22}) = (\varphi(x_{11}) + \varphi(x_{12}))\varphi(a_{22}) = \varphi(x_{11}a_{22}) + \varphi(x_{12}a_{22}) = \varphi(0) + \varphi(x_{12}a_{22}) = \varphi((x_{11} + x_{12})a_{22})$. Hence $za_{22} = (x_{11} + x_{12})a_{22}$. Therefore

$$(z - (x_{11} + x_{12}))a_{22} = 0. \quad (3)$$

Now $\varphi(ze_1) = \varphi(z)\varphi(e_1) = (\varphi(x_{11}) + \varphi(x_{12}))\varphi(e_1) = \varphi(x_{11}e_1) + \varphi(x_{12}e_1) = \varphi(x_{11})$ which implies $ze_1 = x_{11}$. Since $ze_1 = z_{11} + z_{21} = x_{11}$, then $z_{11} = x_{11}$ and $z_{21} = 0$. Hence for all $a_{12} \in \mathfrak{R}_{12}$ we have $\varphi(za_{12}) = \varphi(z)\varphi(a_{12}) = (\varphi(x_{11}) + \varphi(x_{12}))\varphi(a_{12}) = \varphi(x_{11}a_{12}) + \varphi(x_{12}a_{12})$ implying $\varphi((za_{12})e_1) = \varphi(za_{12})\varphi(e_1) = (\varphi(x_{11}a_{12}) + \varphi(x_{12}a_{12}))\varphi(e_1) = \varphi((x_{11}a_{12})e_1) + \varphi((x_{12}a_{12})e_1) = \varphi(x_{12}a_{12})$. It follows that $(za_{12})e_1 = x_{12}a_{12}$, that is,

$$z_{12}a_{12} = x_{12}a_{12}. \quad (4)$$

Thus,

$$(z - (x_{11} + x_{12}))a_{12} = 0, \quad (5)$$

by (4). From identities (1)-(3) and (5) we conclude that

$$(z - (x_{11} + x_{12}))\mathfrak{R} = 0.$$

By condition (ii) of the Theorem, we have $z = x_{11} + x_{12}$. Now assume that $i = k = 1$ and $j = 2$. Again we may find an element z of $\mathfrak{R}$ such that $\varphi(z) = \varphi(x_{11}) + \varphi(x_{21})$. Let us write $z = z_{11} + z_{12} + z_{21} + z_{22}$. For all $a_{11} \in \mathfrak{R}_{11}$ we have $\varphi(a_{11}z) = \varphi(a_{11})\varphi(z) = \varphi(a_{11})(\varphi(x_{11}) + \varphi(x_{21})) = \varphi(a_{11}x_{11}) + \varphi(a_{11}x_{21}) = \varphi(a_{11}x_{11}) + \varphi(0) = \varphi(a_{11}(x_{11} + x_{21}))$. Hence $a_{11}z = a_{11}(x_{11} + x_{21})$, that is,

$$a_{11}(z - (x_{11} + x_{21})) = 0. \quad (6)$$

Next, for all $a_{12} \in \mathfrak{R}_{12}$ we have $\varphi(a_{12}z) = \varphi(a_{12})\varphi(z) = \varphi(a_{12})(\varphi(x_{11}) + \varphi(x_{21})) = \varphi(a_{12}x_{11}) + \varphi(a_{12}x_{21}) = \varphi(0) + \varphi(a_{12}x_{21}) = \varphi(a_{12}(x_{11} + x_{21}))$ implying $a_{12}z = a_{12}(x_{11} + x_{21})$. So

$$a_{12}(z - (x_{11} + x_{21})) = 0. \quad (7)$$

Yet, for all $a_{22} \in \mathfrak{R}_{22}$ we have $\varphi(a_{22}z) = \varphi(a_{22})\varphi(z) = \varphi(a_{22})(\varphi(x_{11}) + \varphi(x_{21})) = \varphi(a_{22}x_{11}) + \varphi(a_{22}x_{21}) = \varphi(0) + \varphi(a_{22}x_{21}) = \varphi(a_{22}(x_{11} + x_{21}))$. Therefore $a_{22}z = a_{22}(x_{11} + x_{21})$, that is,

$$a_{22}(z - (x_{11} + x_{21})) = 0. \quad (8)$$

Now $\varphi(e_1z) = \varphi(e_1)\varphi(z) = \varphi(e_1)(\varphi(x_{11}) + \varphi(x_{21})) = \varphi(e_1x_{11}) + \varphi(e_1x_{21}) = \varphi(x_{11})$ which implies $e_1z = x_{11}$. So $z_{11} = x_{11}$ and $z_{12} = 0$. Hence for all $a_{21} \in \mathfrak{R}_{21}$ we have $\varphi(a_{21}z) = \varphi(a_{21})\varphi(z) = \varphi(a_{21})(\varphi(x_{11}) + \varphi(x_{21})) = \varphi(a_{21}x_{11}) + \varphi(a_{21}x_{21})$ implying $\varphi(e_1(a_{21}z)) = \varphi(e_1)\varphi(a_{21}z) = \varphi(e_1)(\varphi(a_{21}x_{11}) + \varphi(a_{21}x_{21})) = \varphi(e_1(a_{21}x_{11})) + \varphi(e_1(a_{21}x_{21})) = \varphi(a_{21}x_{21})$. It follows that $e_1(a_{21}z) = x_{21}a_{21}$, that is,

$$a_{21}z_{21} = a_{21}x_{21}. \quad (9)$$

Thus

$$a_{21}(z - (x_{11} + x_{21})) = 0, \quad (10)$$

by (9). From identities (6)-(8) and (10) we have

$$\Re(z - (x_{11} + x_{21})) = 0.$$

From condition (iii) of the Theorem we have $z = x_{11} + x_{21}$. Similarly, we prove the cases $i = j = 2; k = 1$ and $i = k = 2; j = 1$. $\square$

Lemma 2.4. $\varphi(x_{12} + x_{21}) = \varphi(x_{12}) + \varphi(x_{21})$.

Proof. Let $z \in \Re$ be such that $\varphi(z) = \varphi(x_{12}) + \varphi(x_{21})$ and let us write $z = z_{11} + z_{12} + z_{21} + z_{22}$. For all $a_{11} \in \Re_{11}$ we have $\varphi(za_{11}) = \varphi(z)\varphi(a_{11}) = (\varphi(x_{12}) + \varphi(x_{21}))\varphi(a_{11}) = \varphi(x_{12}a_{11}) + \varphi(x_{21}a_{11}) = \varphi(0) + \varphi(x_{21}a_{11}) = \varphi((x_{12} + x_{21})a_{11})$. Hence $za_{11} = (x_{11} + x_{12})a_{11}$, that is,

$$(z - (x_{11} + x_{12}))a_{11} = 0. \quad (11)$$

Next, for all $a_{12} \in \Re_{12}$ we have $\varphi(za_{12}) = \varphi(z)\varphi(a_{12}) = (\varphi(x_{12}) + \varphi(x_{21}))\varphi(a_{12}) = \varphi(x_{12}a_{12}) + \varphi(x_{21}a_{12}) = \varphi((x_{12} + x_{21})a_{12})$, by Lemma 2.3. So $za_{12} = (x_{12} + x_{21})a_{12}$. This implies

$$(z - (x_{12} + x_{21}))a_{12} = 0. \quad (12)$$

Also, for all $a_{21} \in \Re_{21}$ we have $\varphi(za_{21}) = \varphi(z)\varphi(a_{21}) = (\varphi(x_{12}) + \varphi(x_{21}))\varphi(a_{21}) = \varphi(x_{12}a_{21}) + \varphi(x_{21}a_{21}) = \varphi((x_{12} + x_{21})a_{21})$, again by Lemma 2.3. Therefore $za_{21} = (x_{12} + x_{21})a_{21}$. Thus

$$(z - (x_{12} + x_{21}))a_{21} = 0. \quad (13)$$

Yet, for all $a_{22} \in \Re_{22}$ we have $\varphi(za_{22}) = \varphi(z)\varphi(a_{22}) = (\varphi(x_{12}) + \varphi(x_{21}))\varphi(a_{22}) = \varphi(x_{12}a_{22}) + \varphi(x_{21}a_{22}) = \varphi(x_{12}a_{22}) + \varphi(0) = \varphi((x_{21} + x_{21})a_{22})$. Therefore $za_{22} = (x_{21} + x_{21})a_{22}$, that is,

$$(z - (x_{21} + x_{21}))a_{22} = 0. \quad (14)$$

From identities (11)-(14), we obtain

$$(z - (x_{12} + x_{21}))\Re = 0$$

which implies $z = x_{12} + x_{21}$, by condition (ii) of the Theorem. $\square$

Lemma 2.5. $\varphi(x_{12} + y_{12}t_{22}) = \varphi(x_{12}) + \varphi(x_{12}t_{22})$.

Proof. First of all, let us note that

$$(e_1 + y_{12})(x_{12} + t_{22}) = x_{12} + y_{12}x_{12} + y_{12}t_{22}.$$

Hence, by Lemmas 2.3 and 2.4 we have $\varphi(x_{12} + y_{12}t_{22}) + \varphi(y_{12}x_{12}) = \varphi(x_{12} + y_{12}t_{22} + y_{12}x_{12}) = \varphi((e_1 + y_{12})(x_{12} + t_{22})) = \varphi(e_1 + y_{12})\varphi(x_{12} + t_{22}) = (\varphi(e_1) + \varphi(y_{12}))(\varphi(x_{12}) + \varphi(t_{22})) = \varphi(x_{12}) + \varphi(y_{12}x_{12}) + \varphi(y_{12}t_{22})$ which results in $\varphi(x_{12} + y_{12}t_{22}) = \varphi(x_{12}) + \varphi(x_{12}t_{22})$. $\square$

Lemma 2.6. $\varphi(x_{21} + y_{12}t_{12}) = \varphi(x_{21}) + \varphi(x_{12}t_{12})$.

Proof. Again, let us note that

$$(x_{21} + y_{12})(e_1 + t_{12}) = x_{21} + x_{21}t_{12} + y_{12}t_{12}.$$

Hence, by Lemmas 2.3 and 2.4 we have $\varphi(x_{21} + y_{12}t_{12}) + \varphi(x_{21}t_{12}) = \varphi(x_{21} + y_{12}t_{12} + x_{21}t_{12}) = \varphi((x_{21} + y_{12})(e_1 + t_{12})) = \varphi(x_{21} + y_{12})\varphi(e_1 + t_{12}) = (\varphi(x_{21}) + \varphi(y_{12}))(\varphi(e_1) + \varphi(t_{12})) = \varphi(x_{21}) + \varphi(x_{21}t_{12}) + \varphi(y_{12}t_{12})$ which results in $\varphi(x_{21} + y_{12}t_{12}) = \varphi(x_{21}) + \varphi(y_{12}t_{12})$. $\square$

Lemma 2.7. $\varphi(x_{11} + y_{12}t_{21}) = \varphi(x_{11}) + \varphi(x_{21}t_{21})$.

Proof. Let us note yet that

$$(x_{11} + y_{12})(e_1 + t_{21}) = x_{11} + y_{12}t_{21}.$$

Hence, by Lemma 2.3 we have $\varphi(x_{11} + y_{12}t_{21}) = \varphi((x_{11} + y_{12})(e_1 + t_{21})) = \varphi(x_{11} + y_{12})\varphi(e_1 + t_{21}) = (\varphi(x_{11}) + \varphi(y_{12}))(\varphi(e_1) + \varphi(t_{21})) = \varphi(x_{11}) + \varphi(y_{12}t_{21})$ which results in $\varphi(x_{11} + y_{12}t_{21}) = \varphi(x_{11}) + \varphi(x_{21}t_{21})$. $\square$

Lemma 2.8. φ is additive on $\mathfrak{R}_{12}$.

Proof. Let $x_{12}, y_{12} \in \mathfrak{R}_{12}$ and $z \in \mathfrak{R}$ such that $\varphi(z) = \varphi(x_{12}) + \varphi(y_{12})$ and let us write $z = z_{11} + z_{12} + z_{21} + z_{22}$. For all $a_{11} \in \mathfrak{R}_{11}$ we have $\varphi(za_{11}) = \varphi(z)\varphi(a_{11}) = (\varphi(x_{12}) + \varphi(y_{12}))\varphi(a_{11}) = \varphi(x_{12}a_{11}) + \varphi(y_{12}a_{11}) = 0$. Therefore $za_{11} = 0 = (x_{12} + y_{12})a_{11}$, that is,

$$(z - (x_{12} + y_{12}))a_{11} = 0. \quad (15)$$

Next, for all $a_{12} \in \mathfrak{R}_{12}$ we have $\varphi(za_{12}) = \varphi(z)\varphi(a_{12}) = (\varphi(x_{12}) + \varphi(y_{12}))\varphi(a_{12}) = \varphi(x_{12}a_{12}) + \varphi(y_{12}a_{12}) = \varphi(x_{12}a_{12} + y_{12}a_{12}) = \varphi((x_{12} + y_{12})a_{12})$, by Lemma 2.6. So $za_{12} = (x_{12} + y_{12})a_{12}$. Therefore

$$(z - (x_{12} + y_{12}))a_{12} = 0. \quad (16)$$

Also, for all $a_{21} \in \mathfrak{R}_{21}$ we have $\varphi(za_{21}) = \varphi(z)\varphi(a_{21}) = (\varphi(x_{12}) + \varphi(y_{12}))\varphi(a_{21}) = \varphi(x_{12}a_{21}) + \varphi(y_{12}a_{21}) = \varphi(x_{12}a_{21} + y_{12}a_{21}) = \varphi((x_{12} + y_{12})a_{21})$, by Lemma 2.7. Hence $za_{21} = (x_{12} + y_{12})a_{21}$. So

$$(z - (x_{12} + y_{12}))a_{21} = 0. \quad (17)$$

Yet, for all $a_{22} \in \mathfrak{R}_{22}$ we have $\varphi(za_{22}) = \varphi(z)\varphi(a_{22}) = (\varphi(x_{12}) + \varphi(y_{12}))\varphi(a_{22}) = \varphi(x_{12}a_{22}) + \varphi(y_{12}a_{22}) = \varphi(x_{12}a_{22} + y_{12}a_{22}) = \varphi((x_{12} + y_{12})a_{22})$, by Lemma 2.5. Therefore $za_{22} = (y_{12} + y_{12})a_{22}$. So

$$(z - (y_{12} + y_{12}))a_{22} = 0. \quad (18)$$

From identities (15)-(18) we have

$$(z - (x_{12} + y_{12}))\mathfrak{R} = 0$$

which implies $z = x_{12} + y_{12}$, by condition (ii) of the Theorem. $\square$

Lemma 2.9. φ is additive on $\mathfrak{R}_{11}$.

Proof. Let $x_{11}, y_{11} \in \mathfrak{R}_{11}$ and $z \in \mathfrak{R}$ such that $\varphi(z) = \varphi(x_{11}) + \varphi(y_{11})$ and let us write $z = z_{11} + z_{12} + z_{21} + z_{22}$. We observe that $\varphi(e_1ze_1) = \varphi(x_{11}) + \varphi(y_{11})$ which implies $\varphi(z_{11}) = \varphi(x_{11}) + \varphi(y_{11})$. Now, for all $r \in \mathfrak{R}$ let us also write $r = r_{11} + r_{12} + r_{21} + r_{22}$. Then $\varphi((e_1ze_1)r_{12}) = \varphi(e_1ze_1)\varphi(r_{12}) = (\varphi(x_{11}) + \varphi(y_{11}))\varphi(r_{12}) = \varphi(x_{11}r_{12}) + \varphi(y_{11}r_{12}) = \varphi(x_{11}r_{12} + y_{11}r_{12}) = \varphi((x_{11} + y_{11})r_{12})$, by Lemma 2.8. Hence $(e_1ze_1)r_{12} = (x_{11} + y_{11})r_{12}$ which implies

$$(e_1(z - (x_{11} + y_{11}))e_1)r_{12} = 0.$$

Therefore

$$(e_1(z - (x_{11} + y_{11}))e_1)(r_{12}e_2) = 0. \quad (19)$$

However, by direct calculi we have

$$(e_1(z - (x_{11} + y_{11}))e_1)(r_{11}e_2) = 0, \quad (20)$$

$$(e_1(z - (x_{11} + y_{11}))e_1)(r_{21}e_2) = 0 \quad (21)$$

and

$$(e_1(z - (x_{11} + y_{11}))e_1)(r_{22}e_2) = 0. \quad (22)$$

From identities (19)-(22) we have

$$(e_1(z - (x_{11} + y_{11}))e_1)(\mathfrak{R}(1 - e_1)) = 0.$$

By condition (iv) of the Theorem we have $e_1(z - (x_{11} + y_{11}))e_1 = 0$, that is, $z_{11} = x_{11} + y_{11}$. Consequently, $\varphi(x_{11} + y_{11}) = \varphi(x_{11}) + \varphi(y_{11})$. $\square$

Lemma 2.10. φ is additive on $e_1\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12}$.

Proof. Let $x_{11}, y_{11} \in \mathfrak{R}_{11}$ and $x_{12}, y_{12} \in \mathfrak{R}_{12}$. By Lemmas 2.3, 2.8 and 2.9 we have $\varphi((x_{11} + x_{12}) + (y_{11} + y_{12})) = \varphi((x_{11} + y_{11}) + (x_{12} + y_{12})) = \varphi(x_{11} + y_{11}) + \varphi(x_{12} + y_{12}) = \varphi(x_{11}) + \varphi(y_{11}) + \varphi(x_{12}) + \varphi(y_{12}) = \varphi(x_{11} + x_{12}) + \varphi(y_{11} + y_{12})$. $\square$

We are ready for proving our main theorem.

Proof. (Proof of Theorem 2.1) Let $x, y \in \mathfrak{R}$ and choose $z \in \mathfrak{R}$ such that $\varphi(z) = \varphi(x) + \varphi(y)$. For each $\alpha \in \Lambda$ let $r \in \mathfrak{R}$ be an arbitrary element. By Lemma 2.10 we see that

$$\begin{aligned} \varphi(e_\alpha(rz)) &= \varphi(e_\alpha)\varphi(rz) = \varphi(e_\alpha)(\varphi(r)\varphi(z)) = \varphi(e_\alpha)(\varphi(r)(\varphi(x) + \varphi(y))) \\ &= \varphi(e_\alpha)(\varphi(r)\varphi(x) + \varphi(r)\varphi(y)) = \varphi(e_\alpha)(\varphi(rx) + \varphi(ry)) \\ &= \varphi(e_\alpha)\varphi(rx) + \varphi(e_\alpha)\varphi(ry) = \varphi(e_\alpha(rx)) + \varphi(e_\alpha(ry)) \\ &= \varphi(e_\alpha(rx) + e_\alpha(ry)) = \varphi(e_\alpha(rx + ry)) = \varphi(e_\alpha(r(x + y))). \end{aligned}$$

which implies $e_\alpha(rz) = e_\alpha(r(x + y))$. Hence, for each $\alpha \in \Lambda$,

$$e_\alpha(\mathfrak{R}(z - (x + y))) = 0.$$

By condition (iii) of the Theorem we have $z = x + y$. $\square$

Corollary 2.11. *Let $\mathfrak{R}$ be an alternative ring containing a family $\{e_\alpha | \alpha \in \Lambda\}$ of idempotents which satisfies:*

- (i) $x\mathfrak{R} = 0$ implies $x = 0$;
- (ii) *If $(e_\alpha\mathfrak{R})x = 0$ (or $e_\alpha(\mathfrak{R}x) = 0$) for each $\alpha \in \Lambda$, then $x = 0$ (and hence $\mathfrak{R}x = 0$ implies $x = 0$);*
- (iii) *For each $\alpha \in \Lambda$, $(e_\alpha x e_\alpha)(\mathfrak{R}(1 - e_\alpha)) = 0$ implies $e_\alpha x e_\alpha = 0$.*

Then any multiplicative isomorphism φ of $\mathfrak{R}$ onto an arbitrary alternative ring is additive.

Corollary 2.12. (Martindale) *Let $\mathfrak{R}$ be an associative ring containing a family $\{e_\alpha | \alpha \in \Lambda\}$ of idempotents with satisfies:*

- (i) $x\mathfrak{R} = 0$ implies $x = 0$;
- (ii) *If $e_\alpha\mathfrak{R}x = 0$ for each $\alpha \in \Lambda$, then $x = 0$ (and hence $\mathfrak{R}x = 0$ implies $x = 0$);*
- (iii) *For each $\alpha \in \Lambda$, $e_\alpha x e_\alpha\mathfrak{R}(1 - e_\alpha) = 0$ implies $e_\alpha x e_\alpha = 0$.*

Then any multiplicative isomorphism φ of $\mathfrak{R}$ onto an arbitrary associative ring is additive.

By [6, §9.2 and §9.3] we have the following definition:

Definition 2.13. *We call an alternative ring $\mathfrak{R}$, with a nonzero center $\mathfrak{Z}$ which does not contain zero divisors of the ring $\mathfrak{R}$, a Cayley-Dickson ring if $\mathfrak{F}$ is the field of quotients of the center $\mathfrak{Z}$, then $\mathfrak{F} \otimes_{\mathfrak{Z}} \mathfrak{R}$ is a Cayley-Dickson algebra.*

Theorem 2.14. *For a 3-torsion free alternative ring $\mathfrak{R}$ the following conditions are equivalent:*

- (i) $\mathfrak{R}$ is prime;
- (ii) $(a\mathfrak{R})b \neq 0$ for any nonzero $a, b \in \mathfrak{R}$;
- (iii) $a(\mathfrak{R}b) \neq 0$ for any nonzero $a, b \in \mathfrak{R}$.

Proof. Clearly all alternative rings satisfying the properties (ii) or (iii) are prime rings. Conversely, suppose that $\mathfrak{R}$ is a 3-torsion free prime alternative ring. If $\mathfrak{R}$ is associative, then the properties (i)-(iii) hold and are equivalents. Now suppose that $\mathfrak{R}$ is not associative ring, then by [5, Lemma 2.4 and Theorem A] $\mathfrak{R}$ is a Cayley-Dickson ring and, by [1, Proposition 1.1], the properties (i)-(iii) hold again and are equivalents. $\square$

As a direct consequence of the Corollary 2.11 and of the Theorem 2.14 we have the following theorem.

Theorem 2.15. *Let $\mathfrak{R}$ be a 3-torsion free prime alternative ring containing a nontrivial idempotent. Then any multiplicative isomorphism φ of $\mathfrak{R}$ onto an arbitrary alternative ring is additive.*

Proof. Let us consider $\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{21} \oplus \mathfrak{R}_{22}$ the Peirce decomposition of $\mathfrak{R}$, relative to a nontrivial idempotent e_1 , and $\Lambda = \{1\}$. Clearly, the properties (i)-(ii) of the Corollary 2.11 hold, by primeness of $\mathfrak{R}$ and the Theorem 2.14 and it will suffice to prove the property (iii). Hence, let x be an element of $\mathfrak{R}$ such that $(e_1 x e_1)(\mathfrak{R} e_2) = 0$ and let us write $x = x_{11} + x_{12} + x_{21} + x_{22}$. Then

$$x_{11} t_{12} = 0, \quad (23)$$

for all $t \in \mathfrak{R}$, where $t = t_{11} + t_{12} + t_{21} + t_{22}$.

Now, let us observe that $\mathfrak{R}_{12} \neq 0$. In fact, if $\mathfrak{R}_{12} = 0$ then $\mathfrak{R}_{21}$ is an ideal of $\mathfrak{R}$. As $\mathfrak{R}_{21}^2 = 0$, then $\mathfrak{R}_{21} = 0$, by primeness of $\mathfrak{R}$. It follows that $\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{22}$ which implies $\mathfrak{R}_{22} = 0$, by primeness of $\mathfrak{R}$ again. So e_1 is a unit element of $\mathfrak{R}$ and this is an absurd.

Thus let s_{12} be a nonzero element of $\mathfrak{R}_{12}$. From direct calculi, using the identity (23) and the Peirce decomposition's properties we have

$$x_{11}(r s_{12}) = 0$$

for all $r \in \mathfrak{R}$. It follows that $x_{11}(\mathfrak{R} s_{12}) = 0$ which implies $x_{11} = 0$, by Theorem 2.14. Consequently, $(e_1 x e_1)(\mathfrak{R}(1 - e_1)) = 0$ implies $e_1 x e_1 = 0$ and the property (iii) is proved. $\square$

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SOME NEW RESULTS ON TIDY GROUPS

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Abstract

Abstract. Given a group G and $x \in G$, the cyclicizer of x is defined to be the subset $Cyc(x) = \{y \in G \mid (x, y) \text{ is cyclic}\}$. The group G is said to be a tidy group if $Cyc(x)$ is a subgroup for all $x \in G$ and a C-tidy group if $Cyc(x)$ is a cyclic subgroup for all $x \in G \setminus K(G)$, where $K(G)$ is the intersection of all the cyclicizers in G . The aim of this paper is to give some new results on finite and infinite tidy (C-tidy) groups.

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1. INTRODUCTION

Given a group G (not necessarily finite) and $x \in G$, the cyclicizer of x is defined to be the subset $Cyc(x) = \{y \in G \mid \langle x, y \rangle \text{ is cyclic}\}$. It is obvious that $Cyc(x) \subseteq C(x)$, where $C(x)$ is the centralizer of x in G . Note that, in general $Cyc(x)$ is not a subgroup of G . For example, in the group $C_2 \times C_4$, we have

$$Cyc((0, 2)) = \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 1), (1, 3)\},$$

which is not a subgroup of $C_2 \times C_4$. The group G is said to be a tidy group if $Cyc(x)$ is a subgroup for all $x \in G$ and a C-tidy group if $Cyc(x)$ is a cyclic subgroup for all $x \in G \setminus K(G)$, where $K(G)$ is the intersection of all the cyclicizers in G . It is easy to see that $K(G)$ is a locally cyclic subgroup of G . We make a convention that a cyclic group is always C-tidy.

Starting with D. Patrick and E. Wepsic in 1991, [9] many authors have studied and characterised groups (finite and infinite) in terms of cyclicizers and tidy properties. In [9], the authors introduced the notion of cyclicizers and studied some of its properties. In [8], the authors introduced the notion of tidy groups and apart from other results they classified finite abelian tidy groups. Using the notion of cyclicizers, the authors in [1, 2] defined non-cyclic graph of a group and studied its influence on a finite group. Moreover, in [6] the authors studied the tidy properties among the finite groups with a partition. On the otherhand, the author in [7] characterised finite groups with at the most eight distinct cyclicizers. Recently, the author in [4, 5], introduced the notion of C-tidy groups and characterised the finite C-tidy groups.

In this paper, among other results we have given some classes of finite and infinite C-tidy groups. Moreover, we have given some necessary and sufficient conditions for a finite tidy group to be a C-tidy group. We have also studied the tidy properties of linear groups of degree two.

Finally, we have studied tidy properties of free and torsion free groups etc. We have used standard group theoretic results for solving our problems.

In this paper, all notations are standard. For example, C_n denotes the cyclic group of order n and C_n^k denotes the direct product of k copies of C_n . The groups $GL(2, q)$, $SL(2, q)$, $PGL(2, q)$ and $PSL(2, q)$ denote the general linear, special linear, projective general linear and the projective special linear groups respectively of degree 2 over the field of size q , where q is a prime power.

2. SOME BASIC RESULTS

It is clear that every subgroup of a tidy (C-tidy) group is tidy (C-tidy). However, the quotient of a tidy (C-tidy) group need not be tidy (C-tidy). For example, free groups are C-tidy (see Corollary 4.4), but the symmetric group S_6 is not tidy, since $(3\ 4\ 5), (3\ 4\ 6) \in Cyc((1\ 2))$ and $(3\ 4\ 5)(3\ 4\ 6) \notin Cyc((1\ 2))$ (note that any group is quotient of a suitable free group). Also we have seen that C_2 and C_4 are tidy (C-tidy), but $C_2 \times C_4$ is not tidy. However, the free product of two tidy (C-tidy) groups is tidy (C-tidy) (see Proposition 4.3).

We begin with some motivating examples of C-tidy groups. By [6, Theorem 3.1], the only tidy groups among the symmetric and the alternating groups are S_m and A_n with $m \leq 5$ and $n \leq 7$. The following proposition shows that the alternating group A_7 is tidy but not C-tidy.

Proposition 2.1. *The only C-tidy groups among the symmetric and the alternating groups are S_m and A_n with $m \leq 5$ and $n \leq 6$.*

Proof. By [6, Theorem 3.1], the only tidy groups among the symmetric and the alternating groups are S_m and A_n with $m \leq 5$ and $n \leq 7$. By some simple calculations, one can verify that the groups S_m and A_n with $m \leq 5$ and $n \leq 6$ are C-tidy. On the otherhand, in A_7 we have $(4\ 5)(6\ 7), (4\ 6)(5\ 7) \in Cyc((1\ 2\ 3))$. Hence A_7 is not C-tidy noting that $(1\ 2\ 3) \in A_7 \setminus K(A_7)$. $\square$

We now classify the finite abelian C-tidy groups. For this we need the following remark which is a consequence of [8, Lemma 10].

Remark 2.2. Let G and H be groups such that $\gcd(|G|, |H|) = 1$. Then $Cyc((g, h)) = Cyc(g) \times Cyc(h)$ for all $(g, h) \in G \times H$.

Proposition 2.3. *Let G be a finite abelian group. Then G is a C-tidy group if and only if G is cyclic or $G \cong C_p^k \times C_m$, where p is a prime and $k \geq 2$, m are integers with $\gcd(p, m) = 1$.*

Proof. Suppose G is a finite abelian C-tidy group. If G is not cyclic, then by [8, Theorem 11], we have $G = C_p^k \times H$, where $k \geq 2$ is an integer and p is a prime with $\gcd(p, |H|) = 1$. Suppose H is not cyclic. Let $a \in C_p^k \setminus K(C_p^k)$ and $b \in K(H)$. By [7, Lemma 2.12], $(a, b) \in G \setminus K(G)$. Now, by Remark 2.2, $\text{Cyc}((a, b)) = C_p \times H$, which is not cyclic. Hence $G \cong C_p^k \times C_m$, where m is an integer with $\gcd(p, m) = 1$. Converse follows from Remark 2.2. $\square$

Proposition 2.4. *The generalized quaternion group Q_{4m} of order $4m$, where $m \geq 2$ is an integer is C-tidy.*

Proof. This group is given by the presentation

$$\langle a, b \mid a^{2m} = 1, b^2 = a^m, b^{-1}ab = a^{-1} \rangle.$$

By a simple calculation, we can see that $\text{Cyc}(a^i b) = \langle a^i b \rangle$, for $0 \leq i \leq 2m-1$. Again, $\text{Cyc}(a) = \langle a \rangle$ and hence Q_{4m} is C-tidy. $\square$

Proposition 2.5. *The dihedral group D_{2n} of order $2n$, where $n \geq 2$ is an integer is C-tidy.*

Proof. This group is given by the presentation

$$\langle a, b \mid a^n = 1 = b^2, b^{-1}ab = a^{-1} \rangle.$$

Note that, if $x \in D_{2n} \setminus \langle a \rangle$, then $o(x) = 2$. Therefore $\text{Cyc}(x) = \langle x \rangle$. On the other hand, if $y \in \langle a \rangle \setminus \{1\}$, then $\text{Cyc}(y) = \langle a \rangle$. Hence the result follows. $\square$

The following proposition gives some necessary and sufficient conditions for a finite tidy group to be a C-tidy group. Note that, given a subgroup H of a group G , one writes $\text{Cyc}(H) = \text{Cyc}_G(H) = \{g \in G \mid \langle g, h \rangle \text{ is cyclic for all } h \in H\}$. Also note that, if G is a finite group and $\langle a, b \rangle$ is cyclic for all $a, b \in G$, then G is cyclic.

Proposition 2.6. *Let G be a finite tidy group. The following are equivalent:*

- (a) G is a C-tidy group,
- (b) Whenever $\langle x, y \rangle$ is cyclic for $x, y \in G \setminus K(G)$, then $\text{Cyc}(x) = \text{Cyc}(y)$,

- (c) If $\langle x, y \rangle$ and $\langle x, z \rangle$ are cyclic, then $\langle y, z \rangle$ is cyclic, whenever $x \in G \setminus K(G)$,
- (d) If U and V are subgroups of G and $K(G) \leq \text{Cyc}(U) \leq \text{Cyc}(V) \leq G$, then $\text{Cyc}(U) = \text{Cyc}(V)$.

Proof. (a) $\implies$ (b)

Let G be a C-tidy group. Suppose $\langle x, y \rangle$ is cyclic for $x, y \in G \setminus K(G)$. Then $\text{Cyc}(x)$ is cyclic and $y \in \text{Cyc}(x)$. Hence $\text{Cyc}(x) \subseteq \text{Cyc}(y)$. Similarly, $\text{Cyc}(y) \subseteq \text{Cyc}(x)$ and hence $\text{Cyc}(x) = \text{Cyc}(y)$.

(b) $\implies$ (c)

If $y \in K(G)$ or $z \in K(G)$, then $\langle y, z \rangle$ is cyclic. Suppose $y, z \in G \setminus K(G)$. Then by (b), $\text{Cyc}(x) = \text{Cyc}(y)$ and $\text{Cyc}(x) = \text{Cyc}(z)$. Thus $\text{Cyc}(y) = \text{Cyc}(z)$ and so $\langle y, z \rangle$ is cyclic.

(c) $\implies$ (d)

Suppose $K(G) \leq \text{Cyc}(U) \leq \text{Cyc}(V) \leq G$. Let $u \in U, v \in V \setminus K(G), x \in \text{Cyc}(U) \setminus K(G)$ and $y \in \text{Cyc}(V) \setminus \text{Cyc}(U)$. Then $\langle x, u \rangle$ is cyclic and $\langle x, v \rangle$ is cyclic. Hence by (c), $\langle u, v \rangle$ is cyclic. Again, $\langle y, v \rangle$ is cyclic and so by (c), $\langle u, y \rangle$ is cyclic. Hence $y \in \text{Cyc}(U)$, which is a contradiction.

(d) $\implies$ (a)

Let $x \in G \setminus K(G)$. Suppose $\text{Cyc}(x)$ is not cyclic. Then there exists $a, b \in \text{Cyc}(x)$ such that $\langle a, b \rangle$ is not cyclic. Then $K(G) \leq \text{Cyc}(\langle x, a \rangle) \leq \text{Cyc}(\langle x \rangle) \leq G$, which contradicts (d). $\square$

The following result gives a necessary and sufficient condition for a finite tidy group to be C-tidy.

Proposition 2.7. *Let G be a finite tidy group. Then G is C-tidy if and only if $\text{Cyc}(x) \cap \text{Cyc}(y) = K(G)$ for all distinct $\text{Cyc}(x)$ and $\text{Cyc}(y)$, where $x, y \in G \setminus K(G)$.*

Proof. Let G be a finite tidy group such that $\text{Cyc}(x) \cap \text{Cyc}(y) = K(G)$ for all distinct $\text{Cyc}(x)$ and $\text{Cyc}(y)$, where $x, y \in G \setminus K(G)$. Let $z \in G \setminus K(G)$. Let $a \in \text{Cyc}(z) \setminus K(G)$. If $\text{Cyc}(a) \neq \text{Cyc}(z)$, then $a \in \text{Cyc}(a) \cap \text{Cyc}(z) = K(G)$, which is a contradiction. Hence we have seen that $\text{Cyc}(a) = \text{Cyc}(z)$ for all $a \in \text{Cyc}(z) \setminus K(G)$. That is $\langle a, b \rangle$ is cyclic for all $a, b \in \text{Cyc}(z) \setminus K(G)$ and hence $\langle a, b \rangle$ is cyclic for all $a, b \in \text{Cyc}(z)$. Therefore $\text{Cyc}(z)$ is cyclic.

Conversely, suppose $Cyc(x) \neq Cyc(y)$. Let $a \in Cyc(x) \cap Cyc(y) \setminus K(G)$. Then $a \in Cyc(x)$ and since $Cyc(x)$ is cyclic, therefore $Cyc(x) \subseteq Cyc(a)$. Similarly, $Cyc(a) \subseteq Cyc(x)$ and hence $Cyc(a) = Cyc(x)$. In the same way, we can show that $Cyc(a) = Cyc(y)$, which is a contradiction. $\square$

A collection Π of non-trivial subgroups of a group G is called a covering if every element of G belongs to a subgroup in Π . The covering Π is called a partition if every non-trivial element of G belongs to a unique subgroup in Π . The subgroups in Π are called the components of Π . In the following result, a subgroup H of G is said to be an R -subgroup if $g \in H$ whenever $g^n \in H \setminus \{1\}$, for some integer n .

Lemma 2.8. *Let G be a group and $\{X_i\}_{i \in I}$ be a covering of G such that $X_i \cap X_j$ is an R -subgroup, for all distinct $i, j \in I$. Then $Cyc(x) \subseteq X_i$ for any $x \in X_i \setminus \{1\}$ and $i \in I$.*

Proof. Let $x \in X_i \setminus \{1\}$ for some $i \in I$. Suppose $y \in Cyc(x) \setminus \{1\}$. Then $\langle y, x \rangle = \langle t \rangle$ for some $t \in G$. Hence $x = t^k$ for some positive integer k . Suppose $t \notin X_i$. Then $t \in X_j$ for some $j \neq i$. Therefore $t^k \in X_i \cap X_j$ and so $t \in X_i$, which is a contradiction. Hence $y \in X_i$. Therefore $Cyc(x) \subseteq X_i$. $\square$

As a consequence we get the following result.

Corollary 2.9. *Let G satisfy the hypothesis of Lemma 2.8. Then G is tidy (C -tidy) if and only if X_i is tidy (C -tidy) for each $i \in I$.*

Proof. Since $Cyc_{X_i}(x) = Cyc_G(x) \cap X_i = Cyc(x)$ for any $x \in X_i \setminus \{1\}$ and $i \in I$, therefore the result follows. $\square$

We would like to conclude this section with a note that, the authors in [6], studied finite tidy groups with a partition. But they did not give any information regarding the tidy property of Frobenius groups and groups of Hughes-Thompson type (see [12] for the classification theorem of finite groups with a partition). In [4, Proposition 2.5], we proved that a finite Frobenius group is tidy if and only if the Frobenius kernel is tidy. Here we consider the groups of Hughes-Thompsons type (not necessarily finite). Recall that, a group G is said to be a group of Hughes-Thompsons type if it is not a p -group and $H_p(G) \neq G$ for some prime p , where $H_p(G) = \langle x \in G \mid o(x) \neq p \rangle$.

Proposition 2.10. *Let G be a group of Hughes-Thompson type. Then $K(G) = \{1\}$, $Cyc(a) \subseteq H_p(G)$ for all $a \in H_p(G) \setminus \{1\}$ and $Cyc(b) = \langle b \rangle$ for all $b \in G \setminus H_p(G)$, p being a prime.*

Proof. Suppose $a \in H_p(G) \setminus \{1\}$ and $y \in Cyc(a)$. Then $\langle a, y \rangle = \langle z \rangle$ for some $z \in G$. If $z \notin H_p(G)$, then $a \in H_p(G) \cap \langle z \rangle = \{1\}$. This contradiction proves $Cyc(a) \subseteq H_p(G)$. Also, if $b \in G \setminus H_p(G)$ and $y \in Cyc(b)$, then $\langle b, y \rangle = \langle z \rangle$ for some $z \in G$. Clearly $z \notin H_p(G)$, which forces $y \in \langle z \rangle = \langle b \rangle$ so that $Cyc(b) = \langle b \rangle$. Finally, $K(G) \subseteq Cyc(a) \cap Cyc(b) = \{1\}$. $\square$

Corollary 2.11. *Let G be a group of Hughes-Thompson type. Then G is tidy if and only if $H_p(G)$ is tidy, p being a prime.*

Proof. Let G be a group of Hughes-Thompson type such that $H_p(G)$ is tidy. By Proposition 2.10, we have $Cyc_G(a) = Cyc_{H_p(G)}(a)$ for all $a \in H_p(G) \setminus \{1\}$ and $Cyc_G(b) = \langle b \rangle$ for all $b \in G \setminus H_p(G)$. Hence G is tidy. Converse is trivial. $\square$

3. TIDY PROPERTY OF LINEAR GROUPS OF DEGREE TWO

The linear groups plays a significant role in the theory of finite groups. In this section we study the tidy (C-tidy) properties of the linear groups of degree two. Let $q = p^n$ be a prime power. In [4], we proved that the groups $G = \text{PGL}(2, q)$ and $G = \text{PSL}(2, q)$ are C-tidy with $K(G) = \{1\}$. In the following results we consider the groups $\text{GL}(2, q)$ and $\text{SL}(2, q)$. We need the following result on $\text{GL}(2, q)$, which is proved in [3].

Proposition 3.1. *Let $G = \text{GL}(2, q)$, where $q > 2$ ($q = p^n$ is a prime power) and a be a non-central element of G . Then $C(a)$ is conjugate to one of the following subgroups:*

- (a) D the subgroup of all diagonal matrices in G . Also $|D| = (q - 1)^2$.
- (b) I a cyclic subgroup of G , where $|I| = q^2 - 1$.
- (c) $PZ(G)$, where P is the Sylow p -subgroup of G containing all matrices as $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$, where x is in the ground field. Also $|PZ(G)| = q(q - 1)$.

Each of the above subgroups is equal to the centralizer of an element in G and G is the union of all conjugates of the above subgroups. Also note that, the number of conjugates of I in G is $q(q - 1)/2$ and number of conjugates of $PZ(G)$ in G is $q + 1$.

We now prove the following result which gives a new characterization of the symmetric group $S_3 (= \text{GL}(2, 2))$. Recall that a group G is said to be a CA-group if $C(x)$ is abelian for all $x \in G \setminus Z(G)$.

Proposition 3.2. *The group $G = \text{GL}(2, q)$, (where $q = p^n$ is a prime power) is tidy (C-tidy) if and only if $q = 2$.*

Proof. Let G be a tidy group and suppose that $q > 2$. By [3, Lemma 3.5], G is a CA-group and hence by [8, Theorem 11], we have P is either cyclic or elementary abelian p -group. We also have $K(G) \subseteq Z(G)$.

Now, suppose that $x \in Z(G) \setminus K(G)$. Note that, for any $y \in P$, we have $\langle x, y \rangle = \langle x \rangle \langle y \rangle$. Therefore $\langle x, y \rangle$ is cyclic as $\gcd(|P|, |Z(G)|) = 1$. It follows that $Cyc(x)$ contains all the conjugates of I as well as all the conjugates of $PZ(G)$. Therefore we have

$$|Cyc(x)| \geq (q^2 - 1) \frac{q(q-1)}{2} + q(q-1)(q+1) + (q-1) - (q-1) \left[\frac{q(q-1)}{2} + (q+1) \right] > \frac{|G|}{2}.$$

Since G is a tidy group, therefore $Cyc(x) = G$, which implies $x \in K(G)$, a contradiction. Therefore we have $K(G) = Z(G)$.

Note that, $Z(G)$ is cyclic. Let $Z(G) = \langle z \rangle$, for some $z \in Z(G) = K(G)$. Let $b \in D \setminus Z(G)$. Then $\langle z, b \rangle$ is cyclic, which implies $b \in Z(G)$, which is again a contradiction, noting that $o(b) \leq q-1$. Therefore $q = 2$. Conversely, if $q = 2$, then $G \cong S_3$, which is clearly tidy (C-tidy). $\square$

We conclude this section with the following result:

Proposition 3.3. *The group $G = \text{SL}(2, q)$ is C-tidy with $K(G) = Z(G)$ for every prime power $q = p^n$.*

Proof. By [6, Theorem 2.6], we have G has a cover $\{X_i\}_{i \in I}$ by cyclic subgroups which intersect pairwise in $Z(G)$. Note that, we have $K(G) \subseteq Z(G)$. Now, if $K(G) \subsetneq Z(G)$, then there exists some $x \in Z(G) \setminus K(G)$ such that $X_i \subseteq Cyc(x)$ for all $i \in I$. But then $Cyc(x) = G$ which is not possible since $x \notin K(G)$. Therefore we have $K(G) = Z(G)$.

Let $a \in G \setminus K(G)$. Then $a \in X_i$ for exactly one $i \in I$. Suppose $b \in Cyc(a)$. Then $\langle a, b \rangle = \langle t \rangle$ for some $t \in G$. Hence $a = t^k$ for some positive integer k . Now, suppose $t \notin X_i$. Then $t \in X_j$ for some $j \in I$, $j \neq i$. But then $a = t^k \in X_i \cap X_j = K(G)$, which is a contradiction. Hence $t \in X_i$ and so $b \in X_i$. It follows that $Cyc(a) \subseteq X_i$ and hence $Cyc(a)$ is cyclic. $\square$

4. TIDY PROPERTIES OF FREE AND TORSION FREE GROUPS

We have already seen that the dihedral group $G = D_{2n}$ of order $2n$ is C-tidy with $K(G) = \{1\}$. The following proposition shows that this result is also true for the infinite dihedral group.

Proposition 4.1. *The infinite dihedral group $G = D_\infty$ is C-tidy with $K(G) = \{1\}$.*

Proof. The infinite dihedral group can be presented in the form

$$D_\infty = \langle a, b \mid a \in Z, b^2 = 1, b^{-1}ab = a^{-1} \rangle.$$

Now, let $x \in D_\infty \setminus \langle a \rangle$. Then one can easily see that $o(x) = 2$. Now, suppose $a^i \in Cyc(x)$ for some $i \in Z$. Then $\langle x, a^i \rangle$ is cyclic and hence $a^i = 1$, otherwise $x \in D_\infty$, which is a contradiction. Therefore $Cyc(x) = \langle x \rangle$. Again $Cyc(a^i) = \langle a \rangle$ for any $a^i \neq 1$. This completes the proof. $\square$

Remark 4.2. In [4, Proposition 2.2], we proved that, if G is a group with a non-trivial partition Π and $X \in \Pi$, then $Cyc(x) \subseteq X$ for any $x (\neq 1) \in X$, and consequently, G is tidy (C-tidy) if and only if every component of Π is tidy (C-tidy). This result is true for any group (finite or infinite) although we considered only finite groups in that paper.

We shall use this result to prove some of our next results.

Proposition 4.3. *Let $G = A * B$ be a free product of groups A and B . Then G is tidy (C-tidy) if and only if A and B are tidy (C-tidy).*

Proof. By the main theorem of [10], we have $\{A^x, B^y, M_z \mid x, y \in G, z \in G \setminus A^x \cup B^y\}$ is a partition of G , where A^x is the conjugate of x in G , B^y is the conjugate of y in G and M_z is the (unique) maximal cyclic subgroup containing z for $z \notin A^x \cup B^y$. Now, the result follows using Remark 4.2. $\square$

Using similar arguments, one can see that the above result is true for a free product of arbitrary many (finite or infinite) number of free factors also. Also, note that free groups are free products of infinite cyclic groups. Therefore as consequence of Proposition 4.3, we have:

Corollary 4.4. *Free groups are C-tidy.*

Given a group G and $s(\neq 1) \in G$, let $G_{2,s}$ be the smallest subgroup of G containing all the $G_{1,(s)}$, where $G_{1,(s)}$ is any cyclic subgroup of G containing s . The following proposition gives a necessary and sufficient condition for a torsion free abelian group to be tidy.

Proposition 4.5. *Let G be a torsion free abelian group. Then G is a tidy group if and only if $G_{2,s} = Cyc(s)$ for any $s \in G \setminus \{1\}$.*

Proof. Let G be a torsion free abelian group which is also a tidy group. By [11, Theorem 5], the collection of all subgroups $G_{2,s}$ as $s(\neq 1)$ ranges over G is a partition of G . One can easily see that $Cyc(s)$ contains all the $G_{1,(s)}$ and hence in view of Remark 4.2, we have $Cyc(s) = G_{2,s}$. Converse is trivial. $\square$

Proposition 4.6. *Let G be a non-cyclic abelian group. Then G is a C-tidy group with $K(G) = \{1\}$ if and only if it satisfies one of the following assertions:*

- (a) G is an elementary abelian p -group for some prime p ,
- (b) G is torsion-free and $G_{2,s}$ is a cyclic subgroup for all $s \in G \setminus \{1\}$.

Proof. Let G be an abelian C-tidy group with $K(G) = \{1\}$. Suppose G has a non-trivial element x of infinite order and an element y of finite order n . We have $(xy)^n = x^n$ and hence $x^n \in \langle xy \rangle$. It follows that $\langle x^n, xy \rangle$ is a cyclic group. Again clearly, $\langle a, a^n \rangle$ is a cyclic group. Using arguments similar to Proposition 2.6, we get $\langle x, xy \rangle$ is cyclic, which is impossible. Therefore G is either a torsion group or a torsion-free group. Also, it is easy to see that the collection of all the proper cyclicizers of G forms a partition of G . Now, if G is a torsion group, then by [11, Theorem 6], G is elementary abelian p group for some prime p and (a) holds. Again, if G is a torsion free group, then (b) holds by Proposition 4.6. Conversely, if (a) holds, then $Cyc(z) = \langle z \rangle$

for any $z(\neq 1) \in G$. On the otherhand, if (b) holds, then the result follows from Proposition 4.6. $\square$

We would like to conclude the paper with the following results, where the product of groups means an arbitrary combination of direct and weak direct product of groups. Also note that, if G is a product of infinite cyclic groups, then G has a partition with cyclic components (see [6, pp. 4]). Therefore by Remark 4.2, G is a C-tidy group.

Proposition 4.7. *Any product of free solvable groups of bounded length is C-tidy.*

Proof. Let G be a product of free solvable groups of bounded length. Using arguments similar to the proof of [6, Theorem 3.2], we have $G = \bigcup_{\alpha \in I} H_{\alpha}$, where H_{α} is C-tidy and $H_{\alpha} \cap H_{\beta} = G'$ is an R-subgroup. Therefore the result follows using Corollary 2.9. $\square$

Proposition 4.8. *Any product of free nilpotent groups of nilpotency class at most 2 is C-tidy.*

Proof. Let G be a product of free nilpotent groups of nilpotency class at most 2. By [6, Theorems 3.2, 3.3], we have $G = \bigcup_{\alpha \in I} H_{\alpha}$, where H_{α} is C-tidy and $H_{\alpha} \cap H_{\beta} = Z(G)$ is an R-subgroup. Therefore the result follows using Corollary 2.9. $\square$

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CATEGORIES OF TOPOLOGICAL ISOSPACES AND ISOGROUPS*

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Abstract

The objects of modern isogeometry and isotopology, like all objects of modern isomathematics, are sets of elements of arbitrary nature endowed with some mathematical isostructures, for example, isosymmetry, etc. The typical way to think about isosymmetry is with the concept of a “isogroup”. But to get a concept of isosymmetry that’s really up to the demands put on it by modern isomathematics, we need — at the very least — to work with a “category” of isosymmetries, rather than a isogroup of isosymmetries. In this article we use Santilli functor for topological isostructures. There are constructed categories of topological vector isospaces, and topological isogroups, and some categorical structures on them.

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1 Introduction

Isomathematics was proposed by Santilli [1] in 1978, and subsequently studied by numerous pure and applied mathematicians as: S. Okubo, H. Myung, M. Tomber, Gr. Tsagas, D. Sourlas, C. Corda, J. Kadeisvili, A. Aringazin, A. Kirhukin, R. Ohemke, G. Wene, G. M. Benkart, J. Osborn, D. Britten, J. Lohmus, E. Paal, L. Sorgsepp, D. Lin, J. Voujouklis, P. Broadbridge, P. Chernoeff, J. Sniatycku, S. Guiasu, E. Prugovecki, A. Sagle, C. Jiang, R. Falcon Ganfornina, J. Nunez Valdes, A. Davvaz, S. G. Georgiev and others.

As a result of these efforts, the new mathematics can be constructed via the systematic application of axiom - preservibg liftings, called isotopies, of the totality of all structures inthe mathematics: including all operators and their operations, icluding the isotopic lifting of numbers, functinal analysis, differential calculus, geometries, topologies, Lie theory and others [2-6].

The physical needs for isomathematics have been indicated in [3,4], and consists in the necessity for a representation of non-Hamiltonian scattering effects in a form that is invariant over time so as to admit the same numerical predictions under the same conditions at different times. Following the study of all possible alternatives, the latter condition required the representation of non-Hamiltonian scattering effects with an axiom-preserving generalization of the trivial (positive-deffinite) unit of quantum mechanics $\hat{h} = 1$ into the most general possible (positive-deffinite as a condition to characterize an isotopy), integro-differential operator $\hat{I}$ which is as positive - definite as +1, functional depending of local variables, that is assumed to be the inverse of the isotopic element $\hat{T}$

$$+1 \hat{\succ} 0 \longrightarrow \hat{I}(t, r, p, a, E, \dots) = \frac{1}{\hat{T}} \hat{\succ} 0$$

and it is called Santilli isounit. Santilli introduced a generalization called lifting of the conventional associative product ab into the form

$$ab \longrightarrow a \hat{\times} b = a \hat{T} b$$

called isoproduct for which:

$$\hat{I} \hat{\times} a = \frac{1}{\hat{T}} \hat{T} a = a \hat{\times} \hat{I} = a \hat{T} \frac{1}{\hat{T}} = a.$$

for every element a of the field of real numbers, complex numbers and quaternions.

The Santilli isonumbers are defined as follows: for given real number or complex number or quaternion a ,

$$\hat{a} = a \hat{I},$$

with isoproduct

$$\hat{a} \hat{\times} \hat{b} = \hat{a} \hat{T} \hat{b} = a \frac{1}{\hat{T}} \hat{T} b \frac{1}{\hat{T}} = ab \frac{1}{\hat{T}} = \hat{ab}.$$

If $a \neq 0$ the corresponding isoelement of $\frac{1}{a}$ will be denoted with $\hat{a}^{-1}$ or $\hat{I} \hat{\times} \hat{a}$.

With $\hat{F}_{\mathbf{R}}$ or $\mathbb{R}$ we will denote the field of iso-reals. Below we will suppose that $\hat{T}_1 \in \hat{F}_{\mathbf{R}}$, $\hat{T}_1 > 0$ is an isotopic element which corresponds of $\hat{F}_{\mathbf{R}}$.

The text is organized and written in a “pedagogical style”, rather than in a highly economical one. The paper is organized into five sections that represent natural “clusters” of topics. Following [13] the second section contains the basic category theory. Following [3–6] in the third section we present basic notions on Isotopies. It also contains more recent research results obtained in [14, 15] in the realm of Santilli functor and concrete categories of isogroups and vector isospaces. In order to make the flow of topics self-motivating, new categorical concepts of isotopology are introduced gradually, by moving from special cases of notions of the category **Top** (with objects all topological spaces and morphisms all continuous functions between them), the category **TopVec** of topological vector spaces and continuous linear transformations and the category **TopGrp** of topological groups as objects and continuous homomorphisms as morphisms to the more general isotopological ones and categorical structures on them.

2 Basic Notions on Categories

Category theory groups together in categories the mathematical objects with some common structure (e.g., sets, partially ordered sets, groups, rings, and so forth) and the appropriate morphisms between such objects [7–13]. These morphisms are required to satisfy certain properties which make the set of all such relations coherent. Given a category, it is not the case that every two objects have a relation between them, some do and others don't. For the ones that do, the number of relations can vary depending on which category we are considering.

DEFINITION 2.1. *A category is a quadruple $(\text{Ob}, \text{Hom}, \text{id}, \circ)$ consisting of:*

- (C1) a class Ob of objects;*
- (C2) for each ordered pair (A, B) of objects a set $\text{Hom}(A, B)$ of morphisms;*
- (C3) for each object A a morphism $\text{id}_A \in \text{Hom}(A, A)$, the identity of A ;*
- (C4) a composition law associating to each pair of morphisms $f \in \text{Hom}(A, B)$ and $g \in \text{Hom}(B, C)$ a morphism $g \circ f \in \text{Hom}(A, C)$;*
which is such that:
 - (M1) $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f \in \text{Hom}(A, B)$, $g \in \text{Hom}(B, C)$ and $h \in \text{Hom}(C, D)$;*
 - (M2) $\text{id}_B \circ f = f \circ \text{id}_A = f$ for all $f \in \text{Hom}(A, B)$;*
 - (M3) the sets $\text{Hom}(A, B)$ are pairwise disjoint.*

This last axiom is necessary so that given a morphism we can identify its domain A and codomain B , however it can always be satisfied by replacing $\text{Hom}(A, B)$ by the set $\text{Hom}(A, B) \times (\{A\}, \{B\})$.

EXAMPLE 2.1. *The classic example is Set , the category with sets as objects and functions as morphisms, and the usual composition of functions as composition.*

2.1 Functors and natural transformations

DEFINITION 2.2. Let $\mathbf{X}$ and $\mathbf{Y}$ be two categories. A **covariant functor** from a category $\mathbf{X}$ to a category $\mathbf{Y}$ is a family of functions $\mathcal{F}$ which associates to each object A in $\mathbf{X}$ an object $\mathcal{F}A$ in $\mathbf{Y}$ and to each morphism $f \in \text{Hom}_{\mathbf{X}}(A, B)$ a morphism $\mathcal{F}f \in \text{Hom}_{\mathbf{Y}}(\mathcal{F}A, \mathcal{F}B)$, and which is such that:

- (F1) $\mathcal{F}(g \circ f) = \mathcal{F}g \circ \mathcal{F}f$ for all $f \in \text{Hom}_{\mathbf{X}}(A, B)$ and $g \in \text{Hom}_{\mathbf{Y}}(B, C)$;
- (F2) $\mathcal{F}\text{id}_A = \text{id}_{\mathcal{F}A}$ for all $A \in \text{Ob}(\mathbf{X})$.

It is clear from the above that a covariant functor is a transformation that preserves both:

- The domains and the codomains identities.
- The composition of arrows, in particular it preserves the direction of the arrows.

DEFINITION 2.3. Let $\mathbf{X}$ and $\mathbf{Y}$ be two categories. A **contravariant functor** from a category $\mathbf{X}$ to a category $\mathbf{Y}$ is a family of functions $\mathcal{F}$ which associates to each object A in $\mathbf{X}$ an object $\mathcal{F}A$ in $\mathbf{Y}$ and to each morphism $f \in \text{Hom}_{\mathbf{X}}(A, B)$ a morphism $\mathcal{F}f \in \text{Hom}_{\mathbf{Y}}(\mathcal{F}A, \mathcal{F}B)$, and which is such that:

- (FI) $\mathcal{F}(g \circ f) = \mathcal{F}f \circ \mathcal{F}g$ for all $f \in \text{Hom}_{\mathbf{X}}(A, B)$ and $g \in \text{Hom}_{\mathbf{Y}}(B, C)$;
- (F2) $\mathcal{F}\text{id}_A = \text{id}_{\mathcal{F}A}$ for all $A \in \text{Ob}(\mathbf{X})$.

Thus, a contravariant functor in mapping arrows from one category to the next reverses the directions of the arrows, by mapping domains to codomains and vice versa. A contravariant functor is also called a presheaf.

So far we have defined categories and maps between them called functors. We will now abstract one step more and define maps between functors. These are called *natural transformations*.

DEFINITION 2.4. Let $\mathcal{F} : \mathbf{X} \rightarrow \mathbf{Y}$ and $\mathcal{G} : \mathbf{X} \rightarrow \mathbf{Y}$ be two functors. A **natural transformation** $\alpha : \mathcal{F} \rightarrow \mathcal{G}$ is given by the following data:

For every object A in $\mathbf{X}$ there is a morphism $\alpha_A : \mathcal{F}(A) \rightarrow \mathcal{G}(A)$ in $\mathbf{Y}$ such that for every morphism $f : A \rightarrow B$ in $\mathbf{X}$ the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\alpha_A} & \mathcal{G}(A) \\ \mathcal{F}(f) \downarrow & & \downarrow \mathcal{G}(f) \\ \mathcal{F}(B) & \xrightarrow{\alpha_B} & \mathcal{G}(B). \end{array}$$

Commutativity means (in terms of equations) that the following compositions of morphisms are equal: $\mathcal{G}(f) \circ \alpha_A = \alpha_B \circ \mathcal{F}(f)$.

The morphisms α_A , $A \in \text{Ob}(\mathbf{X})$, are called the *components of the natural transformation* α .

2.2 Forgetful functor

In a category, two objects x and y can be equal or not equal, but they can be *isomorphic* or not, and if they are isomorphic, they can be isomorphic in many different ways. An isomorphism between x and y is simply a morphism $f : x \rightarrow y$ which has an inverse $g : y \rightarrow x$, such that $f \circ g = \text{id}_y$ and $g \circ f = \text{id}_x$.

In the category **Sets** an isomorphism is just a one-to-one and onto function, i.e. a bijection. If we know two sets x and y are isomorphic we know that they are “the same in a way”, even if they are not equal. But specifying an isomorphism $f : x \rightarrow y$ does more than say x and y are the same in a way; it specifies a *particular way* to regard x and y as the same.

In short, while equality is a yes-or-no matter, a mere *property*, an isomorphism is a *structure*. It is quite typical, as we climb the categorical ladder (here from elements of a set to objects of a category) for properties to be reinterpreted as structures [7–11].

DEFINITION 2.5. We tell that a functor $\mathcal{F} : \mathbf{C} \rightarrow \mathbf{C}'$ define a additional $\mathcal{C}$ -structure on objects of the category $\mathbf{C}'$ if

1. $\forall X, Y \in \text{Ob}(\mathbf{C})$ the map $F : \mathbf{C}(X, Y) \rightarrow \mathbf{C}'(\mathcal{F}X, \mathcal{F}Y)$ is injective,

2. $\forall X \in Ob(\mathbf{C}), Y \in Ob(\mathbf{C}')$ and an isomorphism $u : Y \rightarrow \mathcal{F}(X)$ there is an object $\tilde{Y} \in Ob$ and an isomorphism $\tilde{u} : \tilde{Y} \rightarrow X$ such that $\mathcal{F}(\tilde{Y}) = Y$ and $\mathcal{F}(\tilde{u}) = u$.

Such functor is called a **forgetful functor**.

3 Basic Notions on Isotopies

3.1 Isotopies of the unit and isospaces

The **isothory** is based on the concept of fundamental **isotopy** which is the lifting $I \rightarrow \hat{I}$ of the n -dimensional unit $I = \text{diag}(1, 1, \dots, 1)$ of the Lie's theory into an $n \times n$ -dimensional matrix

$$\hat{I} = (I_j^i) = \hat{I}(t, x, \dot{x}, \ddot{x}, \psi, \psi^+, \partial\psi, \partial\psi^+, \partial\partial\psi, \partial\partial\psi^+, \dots)$$

called the isounit. For simplicity, we consider that maps $I \rightarrow \hat{I}$ are of necessary Kadeisvili Class I (II), the Class III being considered as the union of the first two, i. e. they are sufficiently smooth, bounded, nowhere degenerate, Hermitian and positive (negative) definite, characterizing isotopies (isodualities).

One demands a compatible lifting of all associative products AB of some generic quantities A and B into the isoproduct $A * B$ satisfying the properties:

$$AB \Rightarrow A * B = A\hat{T}B, IA = AI \equiv A \rightarrow \hat{I} * A = A * \hat{I} \equiv A,$$

$$A(BC) = (AB)C \rightarrow A * (B * C) = (A * B) * C,$$

where the fixed and invertible matrix $\hat{T}$ is called the isotopic element.

To follow our outline, a conventional field $F(a, +, \times)$, for instance of real, complex or quaternion numbers, with elements a , conventional sum $+$ and product $a \times b \doteq ab$, must be lifted into the so-called isofield $\hat{F}(\hat{a}, +, *)$, satisfying properties

$$F(a, +, *) \rightarrow \hat{F}(\hat{a}, +, *), \hat{a} = a\hat{I}$$

$$\widehat{a} * \widehat{b} = \widehat{a} \widehat{T} \widehat{b} = (ab) \widehat{I}, \quad \widehat{I} = \widehat{T}^{-1}$$

with elements $\widehat{a}$ called isonumbers, $+$ and $*$ are conventional sum and isoproduct preserving the axioms of the former field $F(a, +, \times)$. All operations in F are generalized for $\widehat{F}$, for instance we have isosquares $\widehat{a}^2 = \widehat{a} * \widehat{a} = \widehat{A} \widehat{T} \widehat{a} = a^2 \widehat{I}$, isoquotient $\widehat{a}/\widehat{b} = (a/b) \widehat{I}$, isosquare roots $\widehat{a}^{1/2} = a^{1/2} \widehat{I}, \dots$; $\widehat{a} * A \equiv aA$. We note that in the literature one uses two types of denotation for isotopic product $*$ or $\widehat{\times}$ (in our work we shall consider $*$ $\equiv$ $\widehat{\times}$).

Let us consider, for example, the main lines of the isotopies of a n -dimensional Euclidean space $E^n(x, g, \mathbb{R})$, where $\mathbb{R}(n, +, \times)$ is the real number field, provided with a local coordinate chart $x = \{x^k\}, k = 1, 2, \dots, n$, and n -dimensional metric $\rho = (\rho_{ij}) = \text{diag}(1, 1, \dots, 1)$. The scalar product of two vectors $x, y \in E^n$ is defined as

$$(x - y)^2 = (x^i - y^i) \rho_{ij} (x^j - y^j) \in \mathbb{R}(n, +, \times)$$

were the Einstein summation rule on repeated indices is assumed hereon.

The Santilli's isoeuclidean spaces $\widehat{E}(\widehat{x}, \widehat{\rho}, \widehat{\mathbb{R}})$ of Class III are introduced as n -dimensional metric spaces defined over an isoreal isofield $\widehat{\mathbb{R}}(\widehat{n}, +, \widehat{\times})$ with an $n \times n$ -dimensional real-valued and symmetrical isounit $\widehat{I} = \widehat{I}^t$ of the same class, equipped with the "isometric"

$$\widehat{\rho}(t, x, v, a, \mu, \tau, \dots) = (\widehat{\rho}_{ij}) = \widehat{T}(t, x, v, a, \mu, \tau, \dots) \times \rho = \widehat{\rho}^t,$$

where $\widehat{I} = \widehat{T}^{-1} = \widehat{I}^t$.

A local coordinate cart on $\widehat{E}(\widehat{x}, \widehat{\rho}, \widehat{\mathbb{R}})$ can be defined in contravariant

$$\widehat{x} = \{\widehat{x}^k = x^{\widehat{k}}\} = \{x^k \times \widehat{I}_k^{\widehat{k}}\}$$

or covariant form

$$\widehat{x}_k = \widehat{\rho}_{kl} \widehat{x}^l = \widehat{T}_k^r \rho_{ri} x^i \times \widehat{I},$$

where $x^k, x_k \in \widehat{E}$. The square of "isoeuclidean distance" between two points $\widehat{x}, \widehat{y} \in \widehat{E}$ is defined as

$$(\widehat{x} - \widehat{y})^2 = [(\widehat{x}^i - \widehat{y}^i) \times \widehat{\rho}_{ij} \times (\widehat{x}^j - \widehat{y}^j)] \times \widehat{I} \in \widehat{R}$$

and the isomultiplication is given by

$$\hat{x}^2 = \hat{x}^k \hat{\times} \hat{x}_k = \left(x^k \times \hat{I} \right) \times \hat{I} \times \left(x_k \times \hat{I} \right) = (x^k \times x_k) \times \hat{I} = n \times \hat{I}.$$

Whenever confusion does not arise isospaces can be practically treated via the conventional coordinates x^k rather than the isotopic ones $\hat{x}^k = x^k \times \hat{I}$. The symbols $x, v, a, \dots$ will be used for conventional spaces while symbols $\hat{x}, \hat{v}, \hat{a}, \dots$ will be used for isospaces; the letter $\hat{\rho}(x, v, a, \dots)$ refers to the projection of the isometric $\hat{\rho}$ in the original space.

We note that an isofield of Class III, explicitly denoted as $\hat{F}_{III}(\hat{a}, +, \hat{\times})$ is a union of two disjoint isofields, one of Class I, $\hat{F}_I(\hat{a}, +, \hat{\times})$, in which the isounit is positive definite, and one of Class II, $\hat{F}_{II}(\hat{a}, +, \hat{\times})$, in which the isounit is negative-definite. The Class II of isofields is usually written as $\hat{F}^d(\hat{a}^d, +, \hat{\times}^d)$ and called isodual fields with isodual unit $\hat{I}^d = -\hat{I} < 0$, isodual isonumbers $\hat{a}^d = a \times \hat{I}^d = -\hat{a}$, isodual isoproduct $\hat{\times}^d = \times \hat{I}^d \times = -\hat{\times}$, etc. For simplicity, in our further considerations we shall use the general terms isofields, isonumbers even for isodual fields, isodual numbers and so on if this will not give rise to ambiguities.

3.2 Isofunctions

An isofunction is a isorelation between a isoset of so - called isoinputs and a isoset of isopermissible so - called isooutputs with the property that each isoinput is isorelated of exactly one isooutput. An example is the isofunction that isorelates each isoreal isonumber $\hat{x}$ to its isosquare $\hat{x} \hat{\times} \hat{x}$. The isooutput of the isofunction $\hat{f}$ corresponding to a isoinput will be denoted with $\hat{f}^\wedge(\hat{x})$.

We will use the notation

$$\hat{f} : \hat{X} \hat{\rightarrow} \hat{Y}$$

and in this context, the isoelements of $\hat{X}$ are called isoarguments of $\hat{f}$, $\hat{X}$ is called isodomain of $\hat{f}$ and $\hat{Y}$ is called isocodomain of $\hat{f}$

DEFINITION 3.1. We will say that the isonumber $\hat{a}$ is isolimit of the isofunction $\hat{f} : \hat{X} \rightarrow \hat{Y}$ at the isopoint $\hat{x}_0$, $\hat{x}_0 \in \hat{X}$, if for every $\hat{\epsilon} \hat{\succ} \hat{0}$ there exists $\hat{\delta} = \hat{\delta}(\hat{\epsilon})$ such that from

$$|\hat{x} - \hat{x}_0| \hat{<} \hat{\delta}$$

follows

$$|\hat{f}^\wedge(\hat{x}) - \hat{a}| \hat{<} \hat{\epsilon}.$$

We will write

$$\hat{f}^\wedge(\hat{x}) \rightarrow_{\hat{x} \rightarrow \hat{x}_0} \hat{a}$$

or

$$\lim_{\hat{x} \rightarrow \hat{x}_0} \hat{f}^\wedge(\hat{x}) = \hat{a}.$$

Let $\hat{f}, \hat{g} : \hat{X} \rightarrow \hat{Y}$, $\hat{x}_0 \in \hat{X}$ and $\lim_{\hat{x} \rightarrow \hat{x}_0} \hat{f}^\wedge(\hat{x}) = \hat{a}$, $\lim_{\hat{x} \rightarrow \hat{x}_0} \hat{g}^\wedge(\hat{x}) = \hat{b}$. Then

1. $\lim_{\hat{x} \rightarrow \hat{x}_0} (\hat{f}^\wedge(\hat{x}) \pm \hat{g}^\wedge(\hat{x})) = \hat{a} \pm \hat{b}$,
2. $\lim_{\hat{x} \rightarrow \hat{x}_0} (\hat{\alpha} \hat{\times} \hat{f}^\wedge(\hat{x})) = \hat{\alpha} \hat{\times} \hat{a}$ for every $\hat{\alpha} \in \hat{F}_K$,
3. $\lim_{\hat{x} \rightarrow \hat{x}_0} (\hat{f}^\wedge(\hat{x}) \hat{\times} \hat{g}^\wedge(\hat{x})) = \hat{a} \hat{\times} \hat{b}$,
4. $\lim_{\hat{x} \rightarrow \hat{x}_0} \hat{f}^\wedge(\hat{x}) \hat{\prec} \hat{g}^\wedge(\hat{x}) = \hat{a} \hat{\prec} \hat{b}$ if $\hat{b} \neq \hat{0}$.

DEFINITION 3.2. An isofunction $\hat{f} : \hat{X} \rightarrow \hat{Y}$ will be called *isocontinuous* at the isopoint $\hat{x}_0 \in \hat{X}$ if for every $\hat{\epsilon} \hat{\succ} \hat{0}$ there exists $\hat{\delta}_1 = \hat{\delta}_1(\hat{\epsilon}) \hat{\succ} \hat{0}$ such that from

$$|\hat{x} - \hat{x}_0| \hat{<} \hat{\delta}_1$$

follows

$$|\hat{f}^\wedge(\hat{x}) - \hat{f}^\wedge(\hat{x}_0)| \hat{<} \hat{\epsilon}.$$

Let $D \subset \mathbb{R}$. With $\hat{F}_D$ we will denote the isoset of the isonumbers $\hat{a}$ for which $a \in D$ and the corresponding basic unit in it is $\hat{I}_1 = \frac{1}{T_1}$, $T_1 \hat{\succ} 0$.

Let $T \in C^1(D)$, $T \hat{\succ} 0$ in D . With $\hat{F}C_D^1$ we will denote the isoset of all isofunctions $\hat{f}$ for which $f \in C^1(D)$ and the corresponding basic unit in it

is $\hat{f} = \frac{1}{T}$, the act of the isofunction $\hat{f} \in \hat{FC}_D^1$ on the isovariable $\hat{x}$ will be denoted as follows

$$\hat{f}^\wedge(\hat{x}) = \hat{f}(T_1\hat{x}) = \frac{f}{T}(T_1x\frac{1}{T_1}) = \frac{f}{T}(x)$$

and if $\hat{g} \in \hat{FC}_D^1$

$$\hat{f} \hat{\times} \hat{g} = \hat{f} T \hat{g}.$$

DEFINITION 3.3. Let $\hat{x} \in \hat{D}$. First isoderivative of the isofunctions $\hat{f} \in \hat{FC}_D^1$ at the isopoint $\hat{x}$ will be called

$$(\hat{f})^\circledast := \lim_{\hat{h} \rightarrow 0} (\hat{f}^\wedge(\hat{x} + \hat{h}) - \hat{f}^\wedge(\hat{x})) \hat{\times} \hat{h}.$$

In this case we will say that $\hat{f}$ is isodifferentiable at the isopoint $\hat{x}$.

$$\begin{aligned} & (\hat{f}^\wedge(\hat{x} + \hat{h}) - \hat{f}^\wedge(\hat{x})) \hat{\times} \hat{h} \\ &= \hat{f}(T_1(\hat{x} + \hat{h})) - \hat{f}(T_1\hat{x}) T_1 \frac{1}{h} \frac{1}{T_1} \\ &= \left(\frac{f}{T}(x + h) - \frac{f}{T}(x) \right) \frac{1}{h} \xrightarrow{h \rightarrow 0} \frac{f'(x)T(x) - f(x)T'(x)}{T^2(x)} \\ &= f'(x) \frac{1}{T(x)} - f(x) \frac{1}{T(x)} \frac{T'(x)}{T(x)} \\ &= \hat{f}' - \hat{f} \hat{\times} \hat{T}' \hat{\times} (\hat{T}) \end{aligned}$$

consequently the representation of the isoderivative in iso-language is

$$(\hat{f})^\circledast = \hat{f}' - \hat{f} \hat{\times} \hat{T}' \hat{\times} (\hat{T}) \quad (1)$$

3.3 Santilli functor

As we have seen the Santilli's central idea [3–5] is the generalization of the fundamental unit of the number theory, from its trivial n -dimensional form $I = \text{diag}(1, 1, \dots)$ to an n -dimensional matrix $\hat{I}$ with the general dependence of all essential variables

$$I = \text{diag}(1, 1, \dots) \Rightarrow \hat{I} = \hat{I}(s, x, \dot{x}, \ddot{x}, \psi, \partial\psi, \partial\partial\psi, \mu, \tau, \dots) \quad (2)$$

under the condition of preserving the original axioms of the unit (nondegeneracy, hermiticity, and positive-definiteness).

The “lifting” $I \Rightarrow \hat{I}$ requires, naturally, for necessary compatibility, a generalization of the conventional associative multiplication $x \circ y$ into the so-called isomultiplication

$$x \circ y \Rightarrow x \hat{\circ} y := xTy, \quad T = \text{fixed}, \quad (3)$$

where the quantity T is called the *isotopic* element. Then $\hat{I} = T^{-1}$ is a correct left and right unit element of the theory with respect the new multiplication $\hat{\circ}$ and it is called the *isounit*.

DEFINITION 3.4. *Let $\mathbf{X}$ and $\mathbf{Y}$ be two categories. A covariant **Santilli functor** from $\mathbf{X}$ to $\mathbf{Y}$ is a family of isofunctions $\hat{I}$ which associates to each object A in $\mathbf{X}$ an object $\hat{I}A$ in $\mathbf{Y}$ and to each morphism $f \in \text{Hom}_{\mathbf{X}}(A, B)$ a morphism $\hat{I}f \in \text{Hom}_{\mathbf{Y}}(\hat{I}A, \hat{I}B)$, and which is such that:*

- (F1) $\hat{I}(g \circ f) = \hat{I}g \circ \hat{I}f$ for all $f \in \text{Hom}_{\mathbf{X}}(A, B)$ and $g \in \text{Hom}_{\mathbf{Y}}(B, C)$;
- (F2) $\hat{I} \text{id}_A = \text{id}_{\hat{I}A}$ for all $A \in \text{Ob}(\mathbf{X})$.

It is clear from the above that a covariant functor is a transformation that preserves both:

- The domains and the codomains identities.
- The composition of arrows, in particular it preserves the direction of the arrows.

DEFINITION 3.5. Let $\mathbf{X}$ and $\mathbf{Y}$ be two categories. A contravariant **Santilli functor** from $\mathbf{X}$ to $\mathbf{Y}$ is a family of isofunctions $\hat{F}$ which associates to each object A in $\mathbf{X}$ an object $\hat{F}A$ in $\mathbf{Y}$ and to each morphism $f \in \text{Hom}_{\mathbf{X}}(A, B)$ a morphism $\hat{F}f \in \text{Hom}_{\mathbf{Y}}(\hat{F}A, \hat{F}B)$, and which is such that:

- (F1) $\hat{F}(g \circ f) = \hat{F}f \circ \hat{F}g$ for all $f \in \text{Hom}_{\mathbf{X}}(A, B)$ and $g \in \text{Hom}_{\mathbf{Y}}(B, C)$;
- (F2) $\hat{F}\text{id}_A = \text{id}_{\hat{F}A}$ for all $A \in \text{Ob}(\mathbf{X})$.

Thus, a contravariant Santilli functor in mapping arrows from one category to the next reverses the directions of the arrows, by mapping domains to codomains and vice versa. A contravariant Santilli functor is also called a Santilli presheaf. These types of Santilli functors will be the principal objects which we will study when discussing Santilli quantum isothory in the language of topos theory.

DEFINITION 3.6. Given two categories C and D , the collection of all covariant (or contravariant) Santilli functors $F : C \rightarrow D$ is actually a category which will be denoted as D^C . This is called the category of Santilli functors and has as objects covariant (or contravariant) Santilli functors and as map natural transformations between Santilli functors.

3.4 Category of Isogroups

One of the simplest algebraic structures is the structure of a group [13]. A set G of elements of any kind is said to be a *group* if a group operation $a \circ b$ is defined in it satisfying the following axioms:

G.1° For any two elements a and b there exists an element

$$c = a \circ b \quad (4)$$

G.2° This operation is associative, that is, for any three elements a, b, c ,

$$(a \circ b) \circ c = a \circ (b \circ c). \quad (5)$$

G.3° There exists a *neutral element* e , i. e. an element such that for every element a ,

$$a \circ e = e \circ a = a. \quad (6)$$

$G.4^\circ$ For each element a there exists a *symmetric element* $\bar{a}$ such that

$$a \circ \bar{a} = \bar{a} \circ a = e. \quad (7)$$

If the group operation $a \circ b$ is called *addition*, we write $c = a + b$ and the element c is called the *sum*, the neutral element is called *zero* and is written as 0, the symmetric element is called the *opposite* and is written as $-a$, and the group is called *additive*.

If the group operation $a \circ b$ is called *multiplication*, we write $c = a \cdot b$, or $c = ab$, the element c is called the *product*, the neutral element is called *unit* and is written as 1, the symmetric element is called the *inverse* and is written as a^{-1} , and the group is called *multiplicative*.

If the group satisfies in addition the axiom

$G.5^\circ$. For any two elements a and b

$$a \circ b = b \circ a, \quad (8)$$

then the group is called *commutative* or *Abelian*.

A set of elements endowed with an operation $a \circ b$ without the properties $G.2^\circ$, $G.3^\circ$ and $G.4^\circ$ is called a *magma*. A magma with the property $G.3^\circ$ is called a *unital magma*, a magma with the property $G.2^\circ$ is called a *semigroup*. A magma in which the equations $a \circ x = b$ and $x \circ a = b$ are solvable for all a and b is called a *quasigroup*. A unital semigroup is called a *monoid*, a unital quasigroup is called a *loop*. All these structures (as also the ones to be introduced yet) are termed infinite, respectively finite, if the underlying set is infinite, respectively finite.

Groups are algebraic systems with one internal composition law. More complicated (and hence, richer) systems are obtained if we introduce a second internal composition law, which is related to the first.

If, in a set of elements of any kind two operations $a+b$ and ab are defined such that

$R.1^\circ$ The set is a commutative group with respect to the operation $a+b$;

$R.2^\circ$ The set is a semigroup with respect to the operation ab ;

R.3° The operation ab is distributive with respect to the operation $a + b$:

$$a(b + c) = ab + ac, \quad (a + b)c = ac + bc, \quad (9)$$

the set is called a *ring*.

A ring in which the set of elements without 0 is a commutative group with respect to the operation ab is called a *field*.

DEFINITION 3.7. A map $f : G \longrightarrow G'$ between two groups $(G, \circ)$ and $(G', \square)$ is called **homomorphism**, if the following property holds:

$$(\forall a, b \in G)[f(a \circ b) = f(a) \square f(b)] \quad (10)$$

Thus a homomorphism "carries" the composition law $\circ$ on G to the composition law $\square$ on G' . Homomorphisms of groups are well visualized in some important aspects with the help of two concepts, the image $\text{Im}(f)$ and the kernel $\text{Ker}(f)$ of the homomorphism.

DEFINITION 3.8. If $f : G \longrightarrow G'$ is a group homomorphism, then we define:

$$a) \text{Im}(f) = \{f(a) \mid a \in G\} \quad (11)$$

$$b) \text{Ker}(f) = \{a \in G \mid f(a) = e' \in G'\}. \quad (12)$$

It is well known that $\text{Im}(f)$ is a subgroup of G' and $\text{Ker}(f)$ is a subgroup of G .

DEFINITION 3.9. A homomorphism f between two groups G and G' is called **isomorphism** if f is bijective. In the case where $G = G'$ an homomorphism f is called **endomorphism** and an isomorphism is called **automorphism**.

DEFINITION 3.10. **Grp** is the category with groups as objects and homomorphisms as morphisms.

DEFINITION 3.11. Let **Grp** and $\widehat{\text{Grp}}$ be two categories.

A Santilli functor $\mathcal{I}$ from **Grp** associates to each object G in **Grp** category

an object $\hat{G}$ in $\widehat{\mathbf{Grp}}$ i.e., we reconstruct the elements for each object $\hat{G}$ of the category $\widehat{\mathbf{Grp}}$ as

$$\mathbf{Grp} \ni a \longrightarrow \hat{a} \equiv a\hat{I} \in \widehat{\mathbf{Grp}}, \quad (13)$$

where the isounit $\hat{I}$ is defined with the help of an invertible element

$$T: \hat{I} = T^{-1}, \quad (14)$$

called isotopic element, and the new composition law is defined by

$$(\forall \hat{a}, \hat{b} \in \widehat{\mathbf{Grp}}) [\hat{a} \hat{\circ} \hat{b} \equiv \hat{a} T \hat{b}]. \quad (15)$$

satisfying the following axioms:

$\hat{G}.1^\circ$ For any two elements $\hat{a}$ and $\hat{b}$ there exists an element

$$\hat{c} = \hat{a} \hat{\circ} \hat{b} \quad (16)$$

$\hat{G}.2^\circ$ This operation is associative, that is, for any three elements $\hat{a}, \hat{b}, \hat{c}$

$$(\hat{a} \hat{\circ} \hat{b}) \hat{\circ} \hat{c} = \hat{a} \hat{\circ} (\hat{b} \hat{\circ} \hat{c}). \quad (17)$$

$\hat{G}.3^\circ$ There exists a isounit $\hat{I}$, i. e. an element such that for every element $\hat{a}$

$$\hat{a} \hat{\circ} \hat{I} \equiv \hat{a} T \hat{I} = \hat{a} T T^{-1} = \hat{a}. \quad (18)$$

$\hat{G}.4^\circ$ For each element $\hat{a}$ there exists a symmetric element $\hat{a}^{-1}$ such that

$$\hat{a} \hat{\circ} \hat{a}^{-1} = \hat{a}^{-1} \hat{\circ} \hat{a} = \hat{I}. \quad (19)$$

If the group satisfies in addition the axiom

$\hat{G}.5^\circ$. For any two elements $\hat{a}$ and $\hat{b}$

$$\hat{a} \hat{\circ} \hat{b} = \hat{b} \hat{\circ} \hat{a}, \quad (20)$$

then the isogroup $\hat{G}$ is called commutative or Abelian.

And the Santilli functor $\mathcal{I}$ from $\mathbf{Grp}$ also associates to each morphism $f \in \text{Hom}_{\mathbf{Grp}}(G, G')$ a morphism $\hat{I}f \in \text{Hom}_{\widehat{\mathbf{Grp}}}(\hat{G}, \hat{G}')$, if the following property holds:

$$(\forall \hat{a}, \hat{b} \in \hat{G}) [\hat{f}(\hat{a} \hat{\circ} \hat{b}) = \hat{f}(\hat{a}) \hat{\square} \hat{f}(\hat{b})]. \quad (21)$$

A map $\hat{f} : \hat{G} \rightarrow \hat{G}'$ between two isogroups $(\hat{G}, \hat{\circ})$ and $(\hat{G}', \hat{\square})$ is called **isohomomorphism**. Thus an isohomomorphism “carries” the composition law $\hat{\circ}$ on $\hat{G}$ to the composition law $\hat{\square}$ on $\hat{G}'$. It can be proved easily, that if $\widehat{\mathbf{Grp}}$ is a monoid and also a groupoid for the fixed isotopic element T , with the above internal composition, it can become a isogroup $\hat{G}$ with unit $\hat{I}$.

DEFINITION 3.12. Let $\mathcal{I} : \mathbf{Grp} \rightarrow \widehat{\mathbf{Grp}}$ and $\mathcal{I}' : \mathbf{Grp} \rightarrow \widehat{\mathbf{Grp}}$ be two Santilli functors. A natural transformation $\hat{\alpha} : \mathcal{I} \rightarrow \mathcal{I}'$ is given by the following data: For every object A in $\mathbf{Grp}$ there is a morphism $\hat{\alpha}_A : \mathcal{I}(A) \rightarrow \mathcal{I}'(A)$ in $\widehat{\mathbf{Grp}}$ such that for every morphism $f : A \rightarrow B$ in $\mathbf{Grp}$ the following diagram is commutative

$$\begin{array}{ccc} \mathcal{I}(A) & \xrightarrow{\hat{\alpha}_A} & \mathcal{I}'(A) \\ \mathcal{I}(f) \downarrow & & \downarrow \mathcal{I}'(f) \\ \mathcal{I}(B) & \xrightarrow{\hat{\alpha}_B} & \mathcal{I}'(B). \end{array}$$

Commutativity means (in terms of equations) that the following compositions of morphisms are equal: $\mathcal{I}(f) \star \hat{\alpha}_A = \hat{\alpha}_B \star \mathcal{I}'(f)$.

The morphisms $\hat{\alpha}_A$, $A \in \text{Ob}(\mathbf{Grp})$, are called the *components of the natural transformation* $\hat{\alpha}$.

3.5 Category of Vector Isospaces

A set L^n of elements of any kind, called *vectors*, is said to be an *n-dimensional vector space* if in this set the operations of *addition* and of *multiplication by scalars*, that is real numbers, are defined, satisfying:

VI.1° - 5° Addition of vectors satisfies axioms G.1–5° for a commutative group;

VII.1° For any vector a and any scalar λ there exists a vector

$$\mathbf{b} = \mathbf{a} \cdot \lambda = \mathbf{a}\lambda \quad (22)$$

called the product of $\mathbf{a}$ by λ ;

VII.2° Multiplication by 1 does not change a vector:

$$\mathbf{a} \cdot 1 = \mathbf{a}; \quad (23)$$

VII.3° Multiplication of vectors by scalars is distributive with respect to addition of scalars:

$$\mathbf{a}(\lambda + \mu) = \mathbf{a}\lambda + \mathbf{a}\mu; \quad (24)$$

VII.4° Multiplication of vectors by scalars is distributive with respect to addition of vectors:

$$(\mathbf{a} + \mathbf{b})\lambda = \mathbf{a}\lambda + \mathbf{b}\lambda; \quad (25)$$

VII.5° Multiplication of vectors by scalars is associative:

$$(\mathbf{a}\lambda)\mu = \mathbf{a}(\lambda\mu); \quad (26)$$

and *axioms VIII.1° - 2° of dimension*, which are based on the notions of linear independence and dependence of vectors. Vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$ are said to be *linearly independent* if a linear combination $\mathbf{a}_1\lambda_1 + \mathbf{a}_2\lambda_2 + \dots + \mathbf{a}_m\lambda_m$ is equal to zero only if all coefficients $\lambda_i = 0$, and *linearly dependent* if there are nonzero coefficients λ_i such that this linear combination is equal to zero.

VIII.1° There exist n linearly independent vectors;

VIII.2° Any $n + 1$ vectors are linearly dependent.

If we have chosen n linearly independent vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ in L_n , then each vector can be written as

$$\mathbf{x} = \sum_i \mathbf{e}_i x^i = \mathbf{e}_i x^i. \quad (27)$$

The numbers x^i are called the *coordinates* of the vector $\mathbf{x}$, the vectors $\mathbf{e}_i$ are called *basis vectors*. Later we will write the sums (27) only in the last form and when in our formulas the same upper and lower indices appear we will always mean summation with respect to these indices.

DEFINITION 3.13. A subset U of a vector space V is called **vector subspace** if it is a subsystem which obeys the axioms of vector space in itself, that is U is closed under vector addition and scalar multiplication.

DEFINITION 3.14. *The notions of $\text{Ker}(f)$ and $\text{Im}(f)$ are defined by the relations*

$$a) \text{Ker}(f) = [x \in V / f(x) = 0 \in U], \quad (28)$$

$$b) \text{Im}(f) = [f(x) \in U / x \in V]. \quad (29)$$

It easy proved that $\text{Ker}(f)$ and $\text{Im}(f)$ are subspaces of V and U respectively.

DEFINITION 3.15. *Let V and U two vector spaces over the same field $\mathbb{F}$ (not necessarily of the same dimension). A map $f : V \longrightarrow U$ is called **linear map** or **linear transformation** if the following property is holds:*

$$(\forall \alpha, \beta \in \mathbb{F}) (\forall x, y \in V) [f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)]. \quad (30)$$

*In case $V = U$, the map f is called **linear operator**.*

Two vector spaces are called *isomorphic* if there is a bijection between them that preserves addition of vectors and multiplication of vectors by scalars, and *homomorphic* if there is a surjection or an injection between them that preserves these operations. Isomorphisms of a vector space onto itself and homomorphisms of a vector space into itself are called *automorphisms* and *endomorphisms* of this vector space respectively. Automorphisms and endomorphisms of a vector space are called *linear transformations* in it.

DEFINITION 3.16. *Vect_k category consists of vector spaces over a field $\mathbb{F}$ as objects and k -linear maps as morphisms.*

From the definition of vector space one can see that we cannot construct an isotopy of a vector space without first introducing an isotopy of the field, because the multiplicative unit I of the space is that of the underlying field. Note that we are lifting of the field, but the elements of the linear space remain unchanged.

DEFINITION 3.17. *Let Vect and $\widehat{\text{Vect}}$ be two categories, V be a vector space over the field $\mathbb{F}$ and $\hat{\mathbb{F}}$ be an isofield of $\mathbb{F}$. A Santilli functor $\mathcal{I}$ from Vect*

associates to each object A in $\mathbf{Vect}$ category an object $\hat{I}A$ in $\widehat{\mathbf{Vect}}$ category by "isovector space" as the vector isospace $\hat{V}$, (which has the same set of the axioms as for a vector space V), over the isofield $\hat{\mathbb{F}}$ equipped with a new external operation $\diamond$ which is such to verify all the axioms for a vector isospace, i.e.,

$\hat{V}I.1^\circ$ - 5° Addition of vectors satisfies axioms $\hat{G}.1-5^\circ$ for a commutative isogroup;

$\hat{V}II.1^\circ$ For any vector $\mathbf{x}$ and any isoscalar $\hat{\alpha}$ there exists a vector

$$(\forall \hat{\alpha} \in \hat{\mathbb{F}}) (\forall \mathbf{x}, \mathbf{y} \in V) [\mathbf{y} = \hat{\alpha} \diamond \mathbf{x}]; \quad (31)$$

called the product of $\mathbf{x}$ by $\hat{\alpha}$;

$\hat{V}II.2^\circ$ Multiplication by $\hat{I}$ does not change a vector:

$$(\forall \mathbf{x} \in V) [\hat{I} \diamond \mathbf{x} = \mathbf{x} \diamond \hat{I} = \mathbf{x}]; \quad (32)$$

$\hat{V}II.3^\circ$ Multiplication of vectors by isoscalars is distributive with respect to addition of isoscalars:

$$(\forall \hat{\alpha}, \hat{\beta} \in \hat{\mathbb{F}}) (\forall \mathbf{x} \in V) [(\hat{\alpha} + \hat{\beta}) \diamond \mathbf{x} = \hat{\alpha} \diamond \mathbf{x} + \hat{\beta} \diamond \mathbf{x}]; \quad (33)$$

$\hat{V}II.4^\circ$ Multiplication of vectors by isoscalars is distributive with respect to addition of isovectors:

$$(\forall \hat{\alpha} \in \hat{\mathbb{F}}) (\forall \mathbf{x}, \mathbf{y} \in V) [\hat{\alpha} \diamond (\mathbf{x} + \mathbf{y}) = \hat{\alpha} \diamond \mathbf{x} + \hat{\alpha} \diamond \mathbf{y}]; \quad (34)$$

$\hat{V}II.5^\circ$ Multiplication of vectors by isoscalars is associative:

$$(\forall \hat{\alpha}, \hat{\beta} \in \hat{\mathbb{F}}) (\forall \mathbf{x} \in V) [\hat{\alpha} \diamond (\hat{\beta} \diamond \mathbf{x}) = (\hat{\alpha} \star \hat{\beta}) \diamond \mathbf{x}]; \quad (35)$$

$\hat{V}III.1^\circ$ There exist n linearly independent isovectors;

$\hat{V}III.2^\circ$ Any $n + 1$ isovectors are linearly dependent.

And the Santilli functor $\mathcal{I}$ from $\mathbf{Vect}$ also associates to each morphism $f \in \text{Hom}_{\mathbf{Vect}}(A, B)$ an continuous linear isotransformation as a morphism $\hat{I}f \in \text{Hom}_{\widehat{\mathbf{Vect}}}(\hat{I}A, \hat{I}B)$:

$$\hat{f}: \hat{V} \longrightarrow \hat{V}', \quad (36)$$

between two vector isospaces $\widehat{V}$ and $\widehat{V}'$ over the same isofield $\widehat{\mathbb{F}}$ which preserves the sum and isomultiplication, i.e., which is such that

$$(\forall \hat{\alpha}, \hat{\beta} \in \widehat{\mathbb{F}}) (\forall \mathbf{x}, \mathbf{y} \in V) [\hat{f}(\hat{\alpha} \diamond \mathbf{x} + \hat{\beta} \diamond \mathbf{y}) = \hat{\alpha} \diamond \hat{f}(\mathbf{x}) + \hat{\beta} \diamond \hat{f}(\mathbf{y})]. \quad (37)$$

4 Category of topological vector isospaces

4.1 Topological vector spaces

A set T of elements is said to be a *topological vector space* if in it subsets called *closed subsets* are singled out and the following axioms are fulfilled:

- $T.1^\circ$ The union of a finite number of closed subsets is closed;
- $T.2^\circ$ The intersection of arbitrary many closed subsets is closed;
- $T.3^\circ$ The whole vector space T is a closed set;
- $T.4^\circ$ The empty set $\emptyset$ is a closed set.

A topological structure also can be defined by means of *open sets*, which are complements of closed sets, by means of *closures of sets* (the closure $\overline{M}$ of a set M is the intersection of all closed sets containing M), by means of *interiors of sets* (the interior of a set M is the union of all open sets contained in M), and by means of *neighborhoods* (i.e. open sets such that any open set can be represented as a union of these sets).

The elements in topological spaces are called *points*, and a neighborhood U containing a point x : is called *neighborhood $U(x)$ of this point*. A point x all neighborhoods $U(x)$ of which contain points of a set M different from x is called a *limit point* of M .

If the set of neighborhoods in a space T is countable, the space is called a *topological vector space with countable base*.

A topological vector space in which the only closed sets are the whole space and the empty set is called a *trivial vector space*. A topological space in which all subsets are closed is called a *discrete vector space*.

The most important topological vector spaces are the Hausdorff vector spaces and regular vector spaces. *Hausdorff vector spaces* satisfy the axioms:

T.5° All points are closed subsets;

T.6° Any two points in the space have disjoint neighborhoods.

Regular vector spaces satisfy the axiom *T.5°* and

T.6' Any point in the vector space and any closed set in this vector space which does not contain this point have disjoint open sets containing this point and this closed set, respectively.

The natural topology in the field $\mathbb{R}$, with closed and open sets defined as in usual real Calculus, can be specified by the countable system of neighborhoods consisting of the intervals with rational ends.

4.2 Categories of topological vector spaces and isospaces and their subcategories

DEFINITION 4.1. *The **TopVec** is the category with topological vector spaces as objects and continuous linear transformations as morphisms.*

The most important subcategories of the category of topological vector spaces **TopVec** are the category of Hausdorff vector spaces **HausVec** and the category of regular vector spaces **RegVec**.

DEFINITION 4.2. *The **HausVec** is the category with Hausdorff vector spaces as objects and continuous linear transformations as morphisms.*

DEFINITION 4.3. *The **RegVec** is the category with regular vector spaces as objects and continuous linear transformations as morphisms.*

The notion of n -dimensional isomanifold was studied by Tsagas and Sourlas (we refer the reader for details in [5]). Their constructions are based on idea that every isounit of Class III can always be diagonalized into the form

$$\hat{I} = \text{diag}(B_1, B_2, \dots, B_n), B_k(x, \dots) \neq 0, k = 1, 2, \dots, n.$$

In result of this one defines a Santilli functor for isotopology $\hat{\tau}$ on $\hat{\mathbb{R}}^n$ which coincides everywhere with the conventional topology τ on $\mathbb{R}^n$ except at the isounit $\hat{I}$.

DEFINITION 4.4. Let $\mathbf{TopVec}$ and $\widehat{\mathbf{TopVec}}$ be two categories, T be a topological vector space over the field $\mathbb{F}$ and $\hat{\mathbb{F}}$ be an isofield of $\mathbb{F}$. A Santilli functor $\mathcal{I}$ from $\mathbf{TopVec}$ associates to each object T in $\mathbf{TopVec}$ category an object $\hat{T}$ in $\widehat{\mathbf{TopVec}}$ category by "isotopological vector space" as the topological vector isospace $\hat{T}$, which satisfy the following axioms:

- $\hat{T}.1^\circ$ The union of a finite number of closed subsets is closed;
- $\hat{T}.2^\circ$ The intersection of arbitrary many closed subsets is closed;
- $\hat{T}.3^\circ$ The whole vector isospace $\hat{T}$ is a closed set;
- $\hat{T}.4^\circ$ The empty set $\emptyset$ is a closed set.

And the Santilli functor $\mathcal{I}$ from $\mathbf{TopVec}$ also associates to each morphism $f \in \mathbf{Hom}_{\mathbf{TopVec}}(A, B)$ an continuous linear isotransformation as a morphism $\hat{f} \in \mathbf{Hom}_{\widehat{\mathbf{TopVec}}}(\hat{A}, \hat{B})$:

$$\hat{f}: \hat{T} \longrightarrow \hat{T}', \quad (38)$$

between two topological vector isospaces $\hat{T}$ and $\hat{T}'$ over the same isofield $\hat{\mathbb{F}}$ which preserves the sum and isomultiplication, i.e., which is such that

$$(\forall \hat{\alpha}, \hat{\beta} \in \hat{\mathbb{F}}) (\forall \mathbf{x}, \mathbf{y} \in T) [\hat{f}(\hat{\alpha} \diamond \mathbf{x} + \hat{\beta} \diamond \mathbf{y}) = \hat{\alpha} \diamond \hat{f}(\mathbf{x}) + \hat{\beta} \diamond \hat{f}(\mathbf{y})]. \quad (39)$$

DEFINITION 4.5. Let $\mathbf{HausVec}$ and $\widehat{\mathbf{HausVec}}$ be two categories, T be a Hausdorff vector space over the field $\mathbb{R}$ and $\hat{\mathbb{R}}$ be an isofield of $\mathbb{R}$. A Santilli functor $\mathcal{I}$ from $\mathbf{HausVec}$ associates to each object T in $\mathbf{HausVec}$ category an object $\hat{T}$ in $\widehat{\mathbf{HausVec}}$ category by "iso-Hausdorff vector space" as the Hausdorff vector isospace $\hat{T}$, which satisfy the following axioms:

- $\widehat{TH}.1^\circ$ The union of a finite number of closed subsets is closed;
 - $\widehat{TH}.2^\circ$ The intersection of arbitrary many closed subsets is closed;
 - $\widehat{TH}.3^\circ$ The whole vector isospace $\hat{T}$ is a closed set;
 - $\widehat{TH}.4^\circ$ The empty set $\emptyset$ is a closed set;
 - $\widehat{TH}.5^\circ$ All points are closed subsets;
 - $\widehat{TH}.6^\circ$ Any two points in the vector isospace have disjoint neighborhoods.
- and the Santilli functor $\mathcal{I}$ from $\mathbf{HausVec}$ also associates to each morphism $f \in \mathbf{Hom}_{\mathbf{HausVec}}(A, B)$ an continuous linear isotransformation as a mor-

phism $\hat{I}f \in \text{Hom}_{\widehat{\text{HausVec}}}(\hat{I}A, \hat{I}B)$:

$$\hat{f} : \hat{T} \longrightarrow \hat{T}', \quad (40)$$

between two Hausdorff vector isospaces $\hat{T}$ and $\hat{T}'$ over the same isofield $\hat{\mathbb{R}}$ which preserves the sum and isomultiplication, i.e., which is such that

$$(\forall \hat{\alpha}, \hat{\beta} \in \hat{\mathbb{R}}) (\forall \mathbf{x}, \mathbf{y} \in T) [\hat{f}(\hat{\alpha} \diamond \mathbf{x} + \hat{\beta} \diamond \mathbf{y}) = \hat{\alpha} \diamond \hat{f}(\mathbf{x}) + \hat{\beta} \diamond \hat{f}(\mathbf{y})]. \quad (41)$$

DEFINITION 4.6. Let RegVec and $\widehat{\text{RegVec}}$ be two categories, T be a regular vector space over the field $\mathbb{R}$ and $\hat{\mathbb{R}}$ be an isofield of $\mathbb{R}$. A Santilli functor $\mathcal{I}$ from RegVec associates to each object T in RegVec category an object $\hat{I}T$ in $\widehat{\text{RegVec}}$ category as the regular vector isospace $\hat{T}$, which satisfy the following axioms:

$\widehat{\text{TR.1}}^\circ$ The union of a finite number of closed subsets is closed;

$\widehat{\text{TR.2}}^\circ$ The intersection of arbitrary many closed subsets is closed;

$\widehat{\text{TR.3}}^\circ$ The whole vector isospace $\hat{T}$ is a closed set;

$\widehat{\text{TR.4}}^\circ$ The empty set $\emptyset$ is a closed set;

$\widehat{\text{TR.5}}^\circ$ All points are closed subsets;

$\widehat{\text{TR.6}}$ Any point in the vector isospace and any closed set in this vector isospace which does not contain this point have disjoint open sets containing this point and this closed set, respectively.

And the Santilli functor $\mathcal{I}$ from RegVec also associates to each morphism $f \in \text{Hom}_{\text{RegVec}}(A, B)$ an continuous linear isotransformation as a morphism $\hat{I}f \in \text{Hom}_{\widehat{\text{RegVec}}}(\hat{I}A, \hat{I}B)$:

$$\hat{f} : \hat{T} \longrightarrow \hat{T}', \quad (42)$$

between two regular vector isospaces $\hat{T}$ and $\hat{T}'$ over the same isofield $\hat{\mathbb{R}}$ which preserves the sum and isomultiplication, i.e., which is such that

$$(\forall \hat{\alpha}, \hat{\beta} \in \hat{\mathbb{R}}) (\forall \mathbf{x}, \mathbf{y} \in T) [\hat{f}(\hat{\alpha} \diamond \mathbf{x} + \hat{\beta} \diamond \mathbf{y}) = \hat{\alpha} \diamond \hat{f}(\mathbf{x}) + \hat{\beta} \diamond \hat{f}(\mathbf{y})]. \quad (43)$$

The natural isotopology in the field $\hat{\mathbb{R}}$, with closed and open sets defined as in usual real iso-Calculus [6], can be specified by the countable system of neighborhoods consisting of the intervals with rational ends.

4.3 Subspaces of topological vector isospaces

A subset of a topological vector isospace where for closed subsets we take the intersections with closed subsets of this space is called a *subisospace*. A topological vector isospace which cannot be divided into two closed non-empty subsets with empty intersection is called *isocontinuous* or *isoconnected*.

A topological vector isospace admitting such a division is called *isonon-connected* and consists of isoconnected components.

A topological vector isospace, or a subset of it, is called *isocompact* if each infinite subset of it has a limit point. In every covering of a compact topological vector isospace we can choose a finite covering.

If $\hat{T}_1, \hat{T}_2, \dots, \hat{T}_n$ are topological vector isospaces, the n -tuples $\{\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n\}$ consisting of points $\hat{x}_i$ in $\hat{T}_i$ form a new topological vector isospace, whose closed sets are sets of such n -tuples for which every point $\hat{x}_i$ runs through a closed subset in $\hat{T}_i$ and arbitrary intersections of these sets.

This new isospace is called the *topological isoproduct* of the topological vector isospaces $\hat{T}_i$.

4.4 Isocontinuous mappings and isohomeomorphisms

A mapping from a topological vector isospace $\hat{T}$ onto a topological vector isospace $\hat{T}'$ is called *isocontinuous* if for each neighborhood $V(\hat{x}')$ of a point $\hat{x}'$ in $\hat{T}'$ there is a neighborhood $U(\hat{x})$ of the corresponding point $\hat{x}$ in $\hat{T}$ such that the images of all points in $U(\hat{x})$ belong to $V(\hat{x}')$.

A bijection from $\hat{T}$ onto $\hat{T}'$ which is isocontinuous together with its inverse bijection is called a *isohomeomorphism*; in this case the topological vector isospaces $\hat{T}$ and $\hat{T}'$ are called *isohomeomorphic*. The sets of closed subsets of two isohomeomorphic topological vector isospaces are mapped onto each other. If $\hat{f}$ is a isocontinuous mapping from $\hat{T}$ to $\hat{T}'$ and all preimages of points of $\hat{T}'$ in $\hat{T}$ are subisospaces isohomeomorphic to a topological vector isospace $\hat{S}$, the space $\hat{T}'$ is called the *quotient isospace* of $\hat{T}$ by $\hat{S}$ and is written as $\hat{T}/\hat{S}$.

4.5 Categorical structures on topological spaces

Among the structures on topological spaces we can select that one, which is compatible with the topology. Let $\mathbf{Top}$ be a category of some topological spaces with a forgetful functor $\mathcal{F} : \mathbf{Top} \rightarrow \mathbf{Set}$.

The categories associated with a topological space $T \in \mathbf{Ob}(\mathbf{Top})$ are as follows:

- The category $\mathbf{T}(T)$, where $\mathbf{Ob}(\mathbf{T})$ is the set of all open subsets of T , and $\mathbf{Mor}(T', T'')$ are all their inclusions.
- The category (pseudogroup) $\mathbf{DE}$, where $\mathbf{Ob}(\mathbf{DE})$ is the set of all open subsets of T , and $\mathbf{Mor}(T', T'')$ are all their homeomorphisms.

Functors $\mathbf{PRESH} : \mathbf{T} \rightarrow \mathbf{Set}$ are called *presheaves* of sets on T . Some of them are called sheaves. Thus we have the inclusions

$$\mathbf{SH}(T) \subset \mathbf{PRESH}(T) \subset \mathbf{FUNCT}(\mathbf{T}, \mathbf{Set}).$$

A *Grothendieck topology* on a category is defined by saying which families of maps into an object constitute a *covering* of the object when certain axioms are fulfilled. A category together with a Grothendieck topology on it is called a *site*. For a site $\mathcal{C}$ one defines the full subcategories $\mathbf{SH}(\mathcal{C}) \subset \mathbf{PRESH}(\mathcal{C}) \subset \mathbf{FUNCT}(\mathcal{C}^\circ, \mathbf{Set})$. The objects of $\mathbf{FUNCT}(\mathcal{C}^\circ, \mathbf{Set})$ are called *presheaves* on the site $\mathcal{C}$, and the objects of $\mathbf{SH}(\mathcal{C})$ are called *sheaves* on $\mathcal{C}$.

For any category there exists the finest topology such that all representable presheaves are sheaves. It is called the *canonical* Grothendieck topology. *Topos* is a category which is equivalent to the category of sheaves for the canonical topology on them.

Hence, the topology is already transferred on a category. So now it is natural to consider on language of toposes and sheaves in all questions connected to local properties.

Here we shall not consider local structures on toposes in general, and we shall restrict ourselves to the consideration of the elementary case of the category **Top**.

DEFINITION 4.7. *A structure defined by a forgetful functor $\mathcal{F} : \mathbf{C} \rightarrow \mathbf{Top}$ is called a **local structure** if*

*$\forall C \in \text{Ob}(\mathbf{C})$ and any inclusion map $i : U \rightarrow \mathcal{F}(C)$ of the open subset U an object $\tilde{U} \in \text{Ob}(\mathbf{C})$ and a morphism $\tilde{i} \in \mathbf{Mor}(\tilde{U}, C)$ exist such that $\mathcal{F}(\tilde{U}) = U$ $\mathcal{F}(\tilde{i}) = i$. This C -structure $\tilde{U}$ is denoted by $C|U$ and called a **restriction of C on U** .*

In other words we can restrict ourselves to local structures on open subsets.

For a local structure $\mathcal{F} : \mathbf{C} \rightarrow \mathbf{Top}$ and each object $X \in \text{Ob}(\mathbf{Top})$ there is the presheaf of categories

$$\mathbf{T}(X)^\circ \rightarrow \mathbf{Cat} : U \mapsto \mathcal{F}^{-1}(U, id_U).$$

Often this presheaf is a sheaf.

5 Category of Topological Isogroups

A set of elements is said to be a *topological group* if

TG.1° This set is a group (see (4)–(7));

TG.2° This set is a topological vector space (see T.1°–T.4°);

TG.3° The group operations $x \rightarrow ax$, $x \rightarrow xa$ and $x \rightarrow x^{-1}$, where a is a constant element, are continuous mappings from this topological space onto itself.

The group and the topological space participating in the definition of a topological group are called the *basis group* and the *basis space* of this topological group. A topological group is called *commutative* or *noncommutative*, *additive* or *multiplicative* together with its basis group, and *connected* or *nonconnected*, *compact* or *noncompact* together with its basis

space. A subset of a topological group is called a *topological subgroup* if it is a subgroup of the basis group and a closed subset of the basis space of the topological group.

DEFINITION 5.1. The **TopGrp** is the category with topological groups as objects and continuous homomorphisms as morphisms.

DEFINITION 5.2. Let **TopGrp** and $\widehat{\text{TopGrp}}$ be two categories.

A Santilli functor $\mathcal{I}$ from **TopGrp** associates to each object G in **TopGrp** category an object $\hat{I}G$ in $\widehat{\text{TopGrp}}$ category as the topological isogroup $\hat{G}$, i.e., we reconstruct a set of elements for each object $\hat{G}$ of the category $\widehat{\text{TopGrp}}$ as A set of elements is said to be a topological group if

$\widehat{TG}.1^\circ$ This set have to be a isogroup (see (16)–(19));

$\widehat{TG}.2^\circ$ This set have to be a topological vector isospace (see $\hat{T}.1^\circ$ – $\hat{T}.4^\circ$);

$\widehat{TG}.3^\circ$ The isogroup operations $\hat{x} \rightarrow \hat{\alpha} \hat{\circ} \hat{x}$, $\hat{x} \rightarrow \hat{x} \hat{\circ} \hat{\alpha}$ and $\hat{x} \rightarrow \hat{x}^{-1}$, where α is a isoscalar, are isocontinuous mappings from this topological vector isospace onto itself.

And the Santilli functor $\mathcal{I}$ from **TopGrp** also associates to each morphism in **TopGrp** a isocontinuous isohomomorphism in $\widehat{\text{TopGrp}}$.

6 Conclusions

In 1996, Santilli generalized in [16] (pages 24-25) the work of Tsagas and Sourlas [5] for the case of isofields of second kind. In 2003, R. M. Falcón and J. Núñez have shown in [17] a possible generalization of Tsagas-Sourlas-Santilli isotopology, by studying the possibility of working with fields that cannot be arranged.

In that last work the notion of isoorder is defined and general notions of (iso)topological isospace are also introduced, to get the definition of isotopology proposed by Tsagas and Sourlas to be a peculiar case of the ones there proposed. We are thinking in a future to make a generalization of Santilli fuctor with isodifferentiable isomanifolds which used in the isodifferential calculus introduced by Santilli in 1996 (see [16]).

That is to say, starting from the generalization of Tsagas-Sourlas-Santilli isotopology give us a tool to build the theory of isocobordism and then the isotopological quantum field theory too.

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**EFFECTIVE DYNAMIC ISO-SPHERE INOPIN
HOLOGRAPHIC RINGS: INQUIRY AND HYPOTHESIS***

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Abstract

In this preliminary work, we focus on a particular iso-geometrical, iso-topological facet of iso-mathematics by suggesting a developing, generalized approach for encoding the states and transitions of spherically-symmetric structures that vary in size. In particular, we introduce the notion of “effective iso-radius” to facilitate a heightened characterization of dynamic iso-sphere Inopin holographic rings (IHR) as they undergo “iso-transitions” between “iso-states”. In essence, we propose the existence of “effective dynamic iso-sphere IHRs”. In turn, this emergence drives the construction of a new “effective iso-state” platform to encode the generalized dynamics of such iso-complex, non-linear systems in a relatively straightforward approach of spherical-based iso-topic liftings. The initial results of this analysis are significant because they lead to alternative modes of research and application, and thereby pose the question: do these effective dynamic iso-sphere IHRs have application in physics and chemistry? Our hypothesis is: yes. To answer this inquiry and assess this conjecture, this developing work should be subjected to further scrutiny, collaboration, improvement, and hard work via the scientific method in order to advance it as such.

Keywords: Geometry and topology; Santilli iso-number; Inopin holographic ring; Iso-radius; Iso-sphere; Dynamic iso-sphere; Effective iso-state.

MSC 2000 numbers: 53A04, 53A05, 54A05, 54A10, 57M50, 57N05, 57N35, 57N37.

*To the memory and honor of Dr. Andrej “Andy” Inopin.

1 Introduction

The new discipline of *Santilli iso-mathematics* [1, 2, 3, 4, 5] has sparked a revolution in the realm of *universal number classification*. More recently, Santilli also generalized his iso-mathematics to create geno-mathematics and hyper-mathematics [1, 2, 3, 4, 5], which are the heart of hadronic mechanics [6, 7] and state-of-the-art, clean, sustainable, industrial-strength energy sources such as MagneGas Fuel [8, 9, 10, 11, 12, 13, 14, 15] and Intermediate Controlled Nuclear Fusion (Synthesis) [16, 17, 18, 19]. This frontier continues to be explored and expanded, where additional efforts have deployed such iso-topic liftings to initiate, for example, new developments toward a *4D topological iso-string theory* [20], *iso-fractals* such as *Mandelbrot iso-sets* [21, 22], and the *iso-dual tesseract* [23]. Furthermore, similar such rigorous examinations of possible iso-mathematics applications have expedited the *dynamic* iso-topic lifting of iso-spaces to install *dynamic iso-spaces* [24], which form the foundation of dynamic iso-sphere IHRs with the “built-in” exterior and interior inverse iso-duality [25].

In this assignment, we focus on advancing the representation of dynamic *iso-1-sphere* IHRs [25] by forging the *effective iso-radius* (“effective iso-modulus” or “effective iso-amplitude-radius”) platform to launch the encoding of their characteristic “iso-transitions” between “iso-states” as they vary in size. The effective iso-radius concept introduced in this paper was originally inspired by the “effective radius” concept from Corda’s new framework of black hole effective states [26, 27, 28, 29, 30]. However, this paper is devoted to iso-mathematics rather than physics; thus, the effective representation proposed here targets spherically-symmetric iso-mathematical structures (like IHRs) rather than spherically-symmetric physical structures (like black holes or hadrons). Hence, for now, we limit our investigation to the domain of iso-mathematics [1, 2, 3, 4, 5] but recognize the hypothesis that such dynamic iso-sphere IHRs may possibly be applied to certain aspects of physics in the future. Thus, we launch our investigation with the step-by-step procedural analysis of Section 2 by presenting a systematic construction of the effective iso-radius for a dynamic iso-1-sphere IHR [25] to initiate the characterization of effective iso-states and iso-transitions for dynamic iso-topic liftings [24]—the new *effective dynamic iso-sphere IHR* is submitted. Finally, we conclude with Section 3, where we recapitulate

the results of Section 2 with a brief discussion and suggest future modes of research.

2 Procedure

In this section, we define and assemble the effective iso-radius for a dynamic iso-sphere IHR [25] and thereby spark the notion of effective iso-state.

2.1 Initializing the 1-sphere IHR topology

Here, we consider the “iso-1-sphere base case” by instantiating the 1-sphere IHR topology [22, 31, 32, 33] via the following procedure:

1. First, from eq. (7) of [22], let $X = \mathbb{C}$ be the set of all complex numbers, the Euclidean complex space, and the *dual 2D Cartesian-polar coordinate-vector state space*, where the complex number $\vec{x} \in X$ is a *dual 2D Cartesian-polar coordinate-vector state* [22, 31, 32, 33] that is defined by eq. (6) of [22] as

$$x = \vec{x} = \vec{x}_{\mathbb{R}} + \vec{x}_{\mathbb{I}} = (\vec{x}) = (|\vec{x}|, \langle \vec{x} \rangle)_P = (\vec{x}_{\mathbb{R}}, \vec{x}_{\mathbb{I}})_C, \quad \forall \vec{x} \in X. \quad (1)$$

In eq. (1), $(\vec{x}_{\mathbb{R}}, \vec{x}_{\mathbb{I}})_C$ is a *2D Cartesian coordinate-vector state* in the *2D Cartesian coordinate-vector state space* X_C so $(\vec{x}_{\mathbb{R}}, \vec{x}_{\mathbb{I}})_C \in X_C$, while $(|\vec{x}|, \langle \vec{x} \rangle)_P$ is a *2D polar coordinate-vector state* in the *2D polar coordinate-vector state space* X_P so $(|\vec{x}|, \langle \vec{x} \rangle)_P \in X_P$, where X_C and X_P are iso-morphic, dual, synchronized, and interlocking in X [22, 31, 32, 33]. Thus, eq. (1) complies with the constraints imposed by eqs. (8–13) of [22]—see Figure 1.

2. Second, from eq. (16) of [22] we have

$$T^1 = \{\vec{x} \in X : |\vec{x}| = r\}, \quad (2)$$

where $T^1 \subset X$ is the 1-sphere IHR of amplitude-radius $r > 0$ (with corresponding curvature $\kappa = \frac{1}{r}$) that is centered on the origin $O \in X$ [22, 31, 32, 33]; T^1 is the multiplicative group of all non-zero complex numbers with amplitude-radius r , which is iso-metrically embedded in X and is simultaneously dual to the two complex sub-spaces X_- and X_+ [22, 31, 32, 33]—see Figure 2.

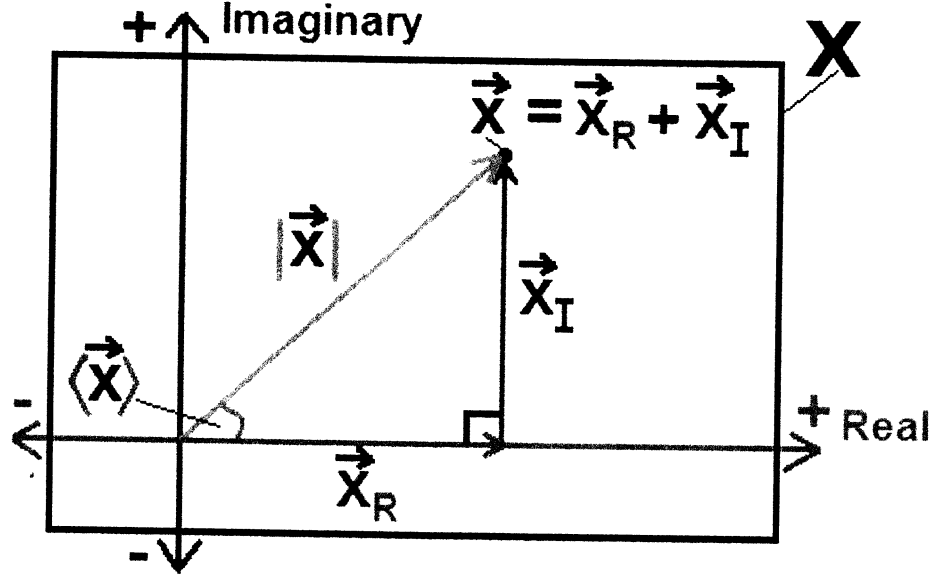


Fig. 1: Complex components for the dual 2D Cartesian-polar coordinate-vector state $\vec{x}$ in the dual 2D Cartesian-polar coordinate-vector state space (and Euclidean complex space) X , such that $\vec{x} \in X$, where $\vec{x}$ is simultaneously treated as a complex number, 2D polar coordinate-vector, and 2D Cartesian coordinate-vector [22, 31, 32, 33]. Specifically, $(\vec{x}_R, \vec{x}_I)_C$ is a 2D Cartesian coordinate-vector state in the 2D Cartesian coordinate-vector state space X_C so $(\vec{x}_R, \vec{x}_I)_C \in X_C$, while $(|\vec{x}|, \langle \vec{x} \rangle)_P$ is a 2D polar coordinate-vector state in the 2D polar coordinate-vector state space X_P so $(|\vec{x}|, \langle \vec{x} \rangle)_P \in X_P$, where X_C and X_P are iso-morphic, dual, synchronized, and interlocking in X [22, 31, 32, 33]. Note that $\vec{x}_R$ and $\vec{x}_I$ are treated as vectors (with axis-dependent magnitude and direction) so the vector sum is $\vec{x} = \vec{x}_R + \vec{x}_I$ with amplitude $|\vec{x}|$ and direction $\langle \vec{x} \rangle$ [22, 31, 32, 33].

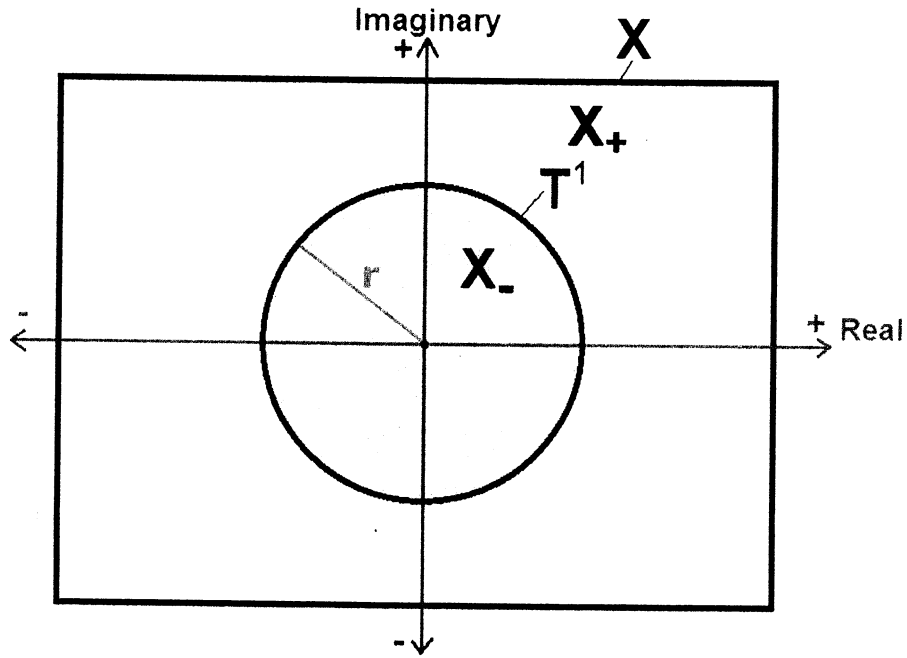


Fig. 2: The 1-sphere IHR topology for the dual 2D Cartesian-polar coordinate-vector state space (and Euclidean complex space) X , where the topological 1-sphere IHR $T^1 \subset X$ is simultaneously dual to two spatial 2-branes [22, 31, 32, 33]: the “2D micro sub-space zone” $X_- \subset X$ and the “2D macro sub-space zone” $X_+ \subset X$ for interior and exterior dynamical systems, respectively [22, 31, 32, 33].

At this point, we've initialized the 1-sphere IHR topology of $T^1 \subset X$ [22, 31, 32, 33]. Therefore, we are ready to explore the proposed the effective iso-radius encoding platform of Section 2.2.

2.2 Constructing the effective iso-radius for the effective iso-state

Here, now that we've initialized the 1-sphere IHR topology of $T^1 \subset X$ [22, 31, 32, 33] in Section 2.1, we are ready to introduce and assemble the effective iso-radius encoding platform for representing iso-sphere IHR [25] iso-states and iso-transitions via the following procedure:

1. First, in conventional mathematics, the number 1 for the multiplicative identity satisfies the original number field axioms [34]. Thus, the number 1 is the fundamental unit that plays important and diverse roles throughout mathematics in general such as, for example, normalization. Therefore, we start by setting the amplitude-radius $r = 1$ so T^1 is the IHR unit-circle with the equivalent curvature $\kappa = \frac{1}{r} = 1$.
2. Second, in iso-mathematics [1, 2, 3, 4, 5, 22], Santilli successfully demonstrated that the multiplicative unit is not limited to the number 1 and can therefore be replaced with the positive-definite iso-multiplicative iso-unit $\hat{r} > 0$ with corresponding inverse $\hat{\kappa} = \frac{1}{\hat{r}} > 0$ for iso-numbers. Hence, for some selected $\hat{r}$, we employ Santilli's isomethodology [1, 2, 3, 4, 5] to iso-topically lift T^1 via

$$\vec{x}_{\hat{r}} \equiv \vec{x} \times \hat{r}, \forall \vec{x} \in T^1 \rightarrow \forall \vec{x}_{\hat{r}} \in T_{\hat{r}}^1, \quad (3)$$

for the transition and its inverse

$$\begin{aligned} f(\hat{r}, T^1) : \quad T^1 &\rightarrow T_{\hat{r}}^1 \\ f^{-1}(\hat{r}, T_{\hat{r}}^1) : \quad T_{\hat{r}}^1 &\rightarrow T^1 \end{aligned} \quad (4)$$

to identify the iso-1-sphere IHR $T_{\hat{r}}^1$ with the iso-radius $\hat{r}$, so T^1 and $T_{\hat{r}}^1$ are *locally iso-morphic* and are both centered on the origin $O \in X$. Here, note that in addition to being the iso-radius of $T_{\hat{r}}^1$, $\hat{r}$ also serves as the iso-unit for Santilli's iso-multiplication [1, 2, 3, 4, 5, 22], where the iso-unit inverse $\hat{\kappa}$ is also the iso-curvature of $T_{\hat{r}}^1$.

3. Third, given the dynamic iso-topic lifting and dynamic iso-spheres of [24, 25], we furthermore define $T_{\hat{r}}^1$'s iso-radius $\hat{r}$ as an iso-function in the positive-definite form

$$T_{\hat{r}(m)}^1 : \hat{r} \equiv \hat{r}(m) \equiv ma + b \equiv \frac{1}{\hat{\kappa}(m)} > 0, \quad (5)$$

where $\hat{r}(m)$ is the *dynamic iso-radius iso-function* (or “dynamic iso-unit iso-function”) with the parameter m and $\hat{\kappa}(m)$ is the corresponding *dynamic iso-curvature iso-function*, such that m is some general *mathematical* quantity while a and b are coefficients that may be customized depending on context and application (i.e. the classic “point-intercept form”, etc.). Thus, eq. (3) is rewritten as

$$\vec{x}_{\hat{r}(m)} \equiv \vec{x} \times \hat{r}(m), \quad \forall \vec{x} \in T^1 \rightarrow \forall \vec{x}_{\hat{r}(m)} \in T_{\hat{r}(m)}^1 \quad (6)$$

so eq. (4) becomes

$$\begin{aligned} f(\hat{r}(m), T^1) : \quad T^1 &\rightarrow T_{\hat{r}(m)}^1 \\ f^{-1}(\hat{r}(m), T_{\hat{r}(m)}^1) : \quad T_{\hat{r}(m)}^1 &\rightarrow T^1. \end{aligned} \quad (7)$$

4. Fourth, given that eq. (5) is a *dynamic iso-unit iso-function*, we wish to show that $\hat{r}(m)$ is characterized by change as it's parameter m varies and takes on values from some positive-definite sequence M , such that $m \in M$ as $m \rightarrow \infty$. In [24, 25], there are *two* distinct types of dynamic iso-unit iso-functions:

- *continuous dynamic iso-unit iso-functions*, so M may be a *continuous* sequence of positive-definite values such as, for example, the case of $M \equiv M_{\mathbb{R}^+}$ for the positive-definite interval of *real numbers*

$$M_{\mathbb{R}^+} = (0, \infty_{\mathbb{R}^+}), \quad m \in M_{\mathbb{R}^+}, \quad m \rightarrow \infty_{\mathbb{R}^+}; \quad (8)$$

and

- *discrete dynamic iso-unit iso-functions*, so M may be a *discrete* sequence of positive-definite values such as, for example, the case of $M \equiv M_{\mathbb{N}}$ for the positive-definite set of *natural numbers*

$$M_{\mathbb{N}} = \{1, 2, 3, 4, 5, \dots\}, \quad m \in M_{\mathbb{N}}, \quad m \rightarrow \infty_{\mathbb{N}} \quad (9)$$

or in the case of $M \equiv M_{Fib}$ for the positive-definite set of *Fibonacci numbers*

$$M_{Fib} = \{1, 1, 2, 3, 5, \dots\}, \quad m \in M_{Fib}, \quad m \rightarrow \infty_{Fib}. \quad (10)$$

5. Fifth, for this introductory investigation, consider a relatively simple case and suppose that $a = 2$ and $b = 0$, where we know that the $\hat{r}(m) > 0$ and $\hat{\kappa}(m) > 0$ of eq. (5) will remain positive-definite as $m > 0$ varies and takes on values from some positive-definite sequence M , regardless of whether M is continuous or discrete. Note: the selection of $a = 2$ is arbitrary and for illustration purposes only (as long as the $\hat{r}(m) > 0$ iso-unit constraint is satisfied), but we are inspired to let $\hat{r}(m) = 2m$ for this example because it parallels the relation between the event horizon radius and mass of Schwarzschild black holes [26, 27, 28, 29, 30]. Thus, eq. (5) is rewritten as

$$T_{\hat{r}(m)}^1 : \hat{r} \equiv \hat{r}(m) \equiv m2 + 0 \equiv 2m \equiv \frac{1}{\hat{\kappa}(m)} > 0. \quad (11)$$

In this example case, we will operate eq. (11) with $\hat{r}(m) = 2m$, but note that eq. (11) could be rewritten again to relate $\hat{r}$ to additional *mathematical* quantities as long as it complies with Santilli's positive-definite iso-unit constraint $\hat{r}(m) > 0$ [1, 2, 3, 4, 5, 22] for the iso-topic liftings of eqs. (6–7).

6. Sixth, in a brief side note, we recall and observe that the fundamental exterior and interior iso-duality inverse establishment [25] gives us

$$\begin{aligned} T_{\hat{r}(m)}^1 &\equiv T_{\hat{r}_+(m)}^1 \\ T_{\hat{\kappa}(m)}^1 &\equiv T_{\hat{r}_-(m)}^1 \end{aligned} \quad (12)$$

because in this context, assuming $\hat{r}(m) > 1$, the $T^1_{\hat{r}(m)} \equiv T^1_{\hat{r}_+(m)}$ of the “outer” iso-radius $\hat{r}(m) \equiv \hat{r}_+(m)$ is the *exterior iso-1-sphere IHR* that is “outside” of T^1 because $T^1_{\hat{r}_+(m)} \subset X_+$, while the $T^1_{\hat{k}(m)} \equiv T^1_{\hat{r}_-(m)}$ of the “inner” iso-radius $\hat{k}(m) \equiv \hat{r}_-(m)$ is the *interior iso-1-sphere IHR* that is “inside” of T^1 because $T^1_{\hat{r}_-(m)} \subset X_-$ [25]: $T^1_{\hat{r}_+(m)}$ and $T^1_{\hat{r}_-(m)}$, or equivalently $T^1_{\hat{r}(m)}$ and $T^1_{\hat{k}(m)}$, are iso-dual [25] due to the fact that

$$\hat{r}_+(m) \equiv \hat{r}(m) \equiv \frac{1}{\hat{k}(m)} \equiv \frac{1}{\hat{r}_-(m)} \quad (13)$$

indicates the inverse property. All of this illustrates a fundamental and important iso-duality between the iso-curvature (the iso-unit inverse) and iso-radius (the iso-unit) when T^1 is iso-topically lifted [25] via

$$T^1_{\hat{k}(m)} \leftarrow T^1 \rightarrow T^1_{\hat{r}(m)}. \quad (14)$$

For a depiction of eqs. (12–14) see Figure 3.

7. Seventh, given eq. (11), we define the *initial iso-radius* as

$$T^1_{\hat{r}(m_0)} : \hat{r}_0 \equiv \hat{r}(m_0) \equiv 2m_0 \equiv \frac{1}{\hat{k}(m_0)} > 0 \quad (15)$$

for the *initial iso-1-sphere IHR iso-state* $T^1_{\hat{r}(m_0)}$, where $\hat{k}(m_0) > 0$ is the *initial iso-curvature*, and $m_0 > 0$ is the *initial quantity*, such that $m_0 \in M$, regardless of whether the positive-definite M is continuous or discrete. Therefore, for this initial case we assign $m = m_0$ for eq. (6) to establish

$$\vec{x}_{\hat{r}(m_0)} \equiv \vec{x} \times \hat{r}(m_0), \forall \vec{x} \in T^1 \rightarrow \forall \vec{x}_{\hat{r}(m_0)} \in T^1_{\hat{r}(m_0)} \quad (16)$$

so eq. (7) becomes

$$\begin{aligned} f(\hat{r}(m_0), T^1) : T^1 &\rightarrow T^1_{\hat{r}(m_0)} \\ f^{-1}(\hat{r}(m_0), T^1_{\hat{r}(m_0)}) : T^1_{\hat{r}(m_0)} &\rightarrow T^1. \end{aligned} \quad (17)$$

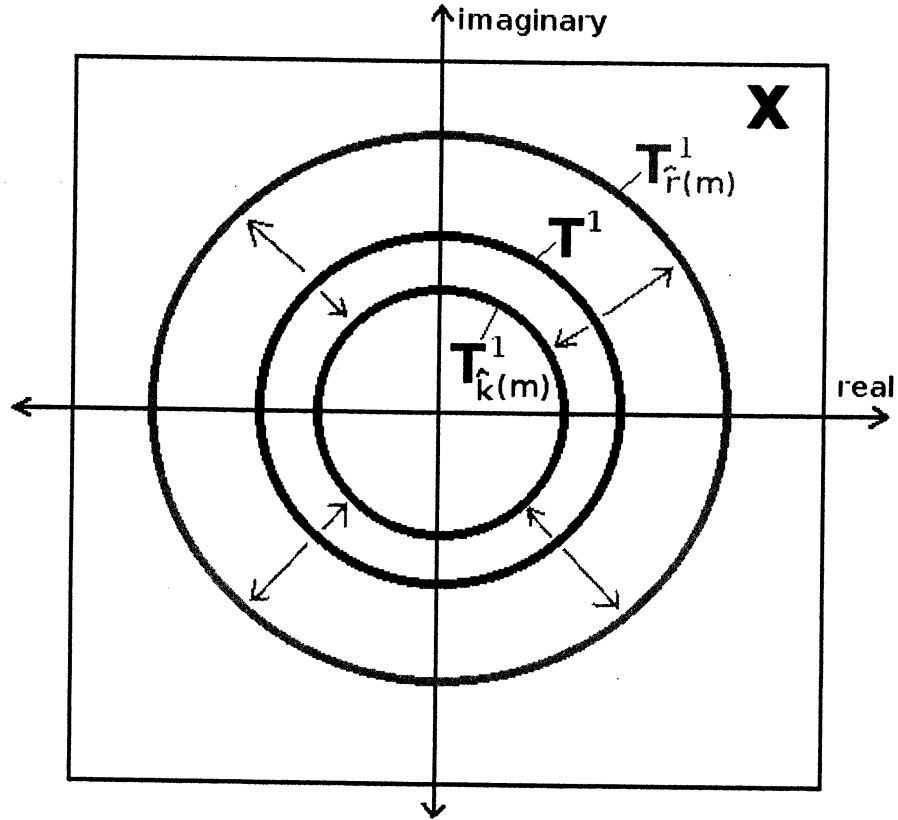


Fig. 3: When the 1-sphere IHR $T^1 \subset X$ of radius $r = 1$ is iso-topically lifted to the iso-1-sphere IHR $T^1_{\hat{r}(m)} = T^1_{\hat{r}_+(m)} \subset X_+$ of iso-radius $\hat{r}(m) = \hat{r}_+(m) > r$ and iso-curvature $\hat{k}(m) = \frac{1}{\hat{r}(m)} = \hat{r}_-(m) < r$, there also exists the iso-1-sphere IHR $T^1_{\hat{k}(m)} = T^1_{\hat{r}_-(m)} \subset X_-$ of iso-radius $\hat{r}_-(m)$ and iso-curvature $\hat{r}_+(m)$, such that $T^1_{\hat{r}_+(m)}$ and $T^1_{\hat{r}_-(m)}$ are iso-dual inverses [25].

8. Eighth, suppose that the quantity m_0 undergoes a change that is characterized by

$$\delta_m : m_0 \rightarrow m_1, \quad (18)$$

which causes

$$\delta_{\hat{r}(m)} : \hat{r}(m_0) \rightarrow \hat{r}(m_1), \quad (19)$$

such that

$$m_0 = m_1 - \Delta_m. \quad (20)$$

Thus, a second version of eq. (15) is written to define the *final iso-radius* as

$$T_{\hat{r}(m_1)}^1 : \hat{r}_1 \equiv \hat{r}(m_1) \equiv 2m_1 \equiv \frac{1}{\hat{\kappa}(m_1)} > 0 \quad (21)$$

for the *final iso-1-sphere IHR iso-state* of $T_{\hat{r}(m_1)}^1$, where $\hat{\kappa}(m_1) > 0$ is the *final iso-curvature*, and $m_1 > 0$ is the *final quantity*, such that $m_1 \in M$, regardless of whether the positive-definite M is continuous or discrete. Therefore, for this final case we assign $m = m_1$ for eq. (6) to establish

$$\vec{x}_{\hat{r}(m_1)} \equiv \vec{x} \times \hat{r}(m_1), \quad \forall \vec{x} \in T^1 \rightarrow \forall \vec{x}_{\hat{r}(m_1)} \in T_{\hat{r}(m_1)}^1 \quad (22)$$

so eq. (17) becomes

$$\begin{aligned} f(\hat{r}(m_1), T^1) : \quad T^1 &\rightarrow T_{\hat{r}(m_1)}^1 \\ f^{-1}(\hat{r}(m_1), T_{\hat{r}(m_1)}^1) : \quad T_{\hat{r}(m_1)}^1 &\rightarrow T^1. \end{aligned} \quad (23)$$

9. Ninth, given the impact of eqs. (18–23), the initial iso-1-sphere IHR iso-state of $T_{\hat{r}(m_0)}^1$ (of initial iso-radius $\hat{r}(m_0)$) is iso-topically lifted to the final iso-1-sphere IHR iso-state of $T_{\hat{r}(m_1)}^1$ (of final iso-radius $\hat{r}(m_1)$) via

$$\vec{x}_{\hat{r}(m_1)} \equiv \vec{x}_{\hat{r}(m_0)} \times \frac{\hat{r}(m_1)}{\hat{r}(m_0)}, \quad \forall \vec{x}_{\hat{r}(m_0)} \in T_{\hat{r}(m_0)}^1 \rightarrow \forall \vec{x}_{\hat{r}(m_1)} \in T_{\hat{r}(m_1)}^1 \quad (24)$$

for the iso-transition and its inverse

$$\begin{aligned} f\left(\frac{\hat{r}(m_1)}{\hat{r}(m_0)}, T_{\hat{r}(m_0)}^1\right) : \quad T_{\hat{r}(m_0)}^1 &\rightarrow T_{\hat{r}(m_1)}^1 \\ f^{-1}\left(\frac{\hat{r}(m_1)}{\hat{r}(m_0)}, T_{\hat{r}(m_1)}^1\right) : \quad T_{\hat{r}(m_1)}^1 &\rightarrow T_{\hat{r}(m_0)}^1 \end{aligned} \quad (25)$$

with the iso-radius ratio $\frac{\hat{r}(m_1)}{\hat{r}(m_0)}$ and the corresponding iso-curvature ratio $\frac{\hat{r}(m_0)}{\hat{r}(m_1)}$ characterize the iso-transition to establish that T^1 , $T^1_{\hat{r}(m_0)}$, and $T^1_{\hat{r}(m_1)}$ are indeed locally iso-morphic—see Figure 4.

10. Tenth, we note that the iso-transition between $T^1_{\hat{r}(m_0)}$ and $T^1_{\hat{r}(m_1)}$ depends on Δ_m and complies with the trichotomy:
 - **Case $\Delta_m < 0$:** $T^1_{\hat{r}(m_0)}$ is *de-magnified* to become $T^1_{\hat{r}(m_1)}$ via the iso-topic lifting $T^1_{\hat{r}(m_0)} \rightarrow T^1_{\hat{r}(m_1)}$ because $m_1 < m_0$ so $\hat{r}(m_1) < \hat{r}(m_0)$.
 - **Case $\Delta_m = 0$:** $T^1_{\hat{r}(m_0)}$ is equivalent to $T^1_{\hat{r}(m_1)}$ via the iso-topic lifting $T^1_{\hat{r}(m_0)} \rightarrow T^1_{\hat{r}(m_1)}$ because $m_1 = m_0$ so $\hat{r}(m_1) = \hat{r}(m_0)$.
 - **Case $\Delta_m > 0$:** $T^1_{\hat{r}(m_0)}$ is *magnified* to become $T^1_{\hat{r}(m_1)}$ via the iso-topic lifting $T^1_{\hat{r}(m_0)} \rightarrow T^1_{\hat{r}(m_1)}$ because $m_1 > m_0$ so $\hat{r}(m_1) > \hat{r}(m_0)$.
11. Finally, given the new and developing framework of [26, 27, 28, 29, 30] that characterizes the effective *physical* state of black holes for an emission or absorption transition, we are motivated to define the effective *iso-mathematical* iso-states of dynamic iso-1-sphere IHRs (which are also spherically-symmetric objects) for a transition from $T^1_{\hat{r}(m_0)}$ to $T^1_{\hat{r}(m_1)}$. Therefore, given the *physical black hole effective radius* definition from eq. (5) of [29], we implement the dynamic iso-topic lifting of [24, 25] and define the (*iso-mathematical*) *effective dynamic iso-1-sphere IHR iso-radius* as

$$T^1_{\hat{r}(m_0)} \rightarrow T^1_{\hat{r}(m_1)} : \hat{r}_E \equiv \hat{r}_E(m_0, m_1) \equiv 2m_E(m_0, m_1) \equiv \frac{1}{\hat{\kappa}_E(m_0, m_1)} > 0, \quad (26)$$

where $\hat{\kappa}_E(m_0, m_1)$ is the *effective dynamic iso-1-sphere IHR iso-curvature* and inverse of the iso-unit, such that the *effective iso-1-sphere IHR quantity* is defined as

$$m_E(m_0, m_1) \equiv \frac{m_0 + m_1}{2}, \quad (27)$$

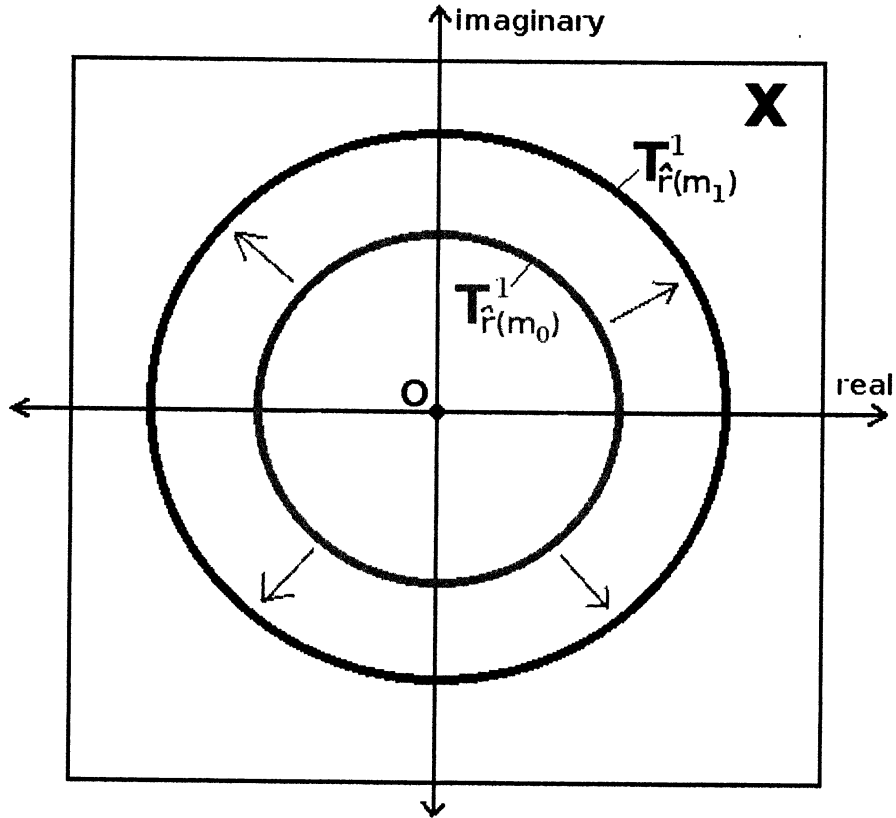


Fig. 4: For the iso-transition $T_{\hat{r}(m_0)}^1 \rightarrow T_{\hat{r}(m_1)}^1$ of Δ_m , the initial dynamic iso-1-sphere IHR iso-state $T_{\hat{r}(m_0)}^1$ is iso-topically lifted to the final dynamic iso-1-sphere IHR iso-state $T_{\hat{r}(m_1)}^1$, where $T_{\hat{r}(m_0)}^1$, and $T_{\hat{r}(m_1)}^1$ are indeed locally iso-morphic.

which is simply the *average* of $T_{\hat{r}(m_0)}^1$'s initial quantity m_0 and $T_{\hat{r}(m_1)}^1$'s final quantity m_1 . Thereafter, we define the *effective dynamic iso-1-sphere IHR* as

$$T_{\hat{r}_E} \equiv T_{\hat{r}_E(m_0, m_1)} \equiv \{\vec{x} \in X : |\vec{x}| = \hat{r}_E(m_0, m_1)\}, \quad (28)$$

where $T_{\hat{r}_E(m_0, m_1)} \subset X$ is centered on the origin $O \in X$; $T_{\hat{r}_E(m_0, m_1)}$ is the multiplicative group of all non-zero complex numbers with the effective iso-amplitude-radius $\hat{r}_E(m_0, m_1)$ —see Figure 5.

At this point, we've assembled a preliminary construction of the effective iso-radius encoding platform for representing dynamic iso-sphere IHR [25] iso-states and iso-transitions for the IHR topology [22, 31, 32, 33].

3 Conclusion

In this work, we merged pertinent aspects from Inopin's IHR topology [22, 31, 32, 33], Santilli's iso-topic liftings [1, 2, 3, 4, 5] and Corda's effective radius [26, 27, 28, 29, 30] into a single iso-mathematical model of effective dynamic iso-sphere IHRs [25] with effective iso-radii. More specifically, we successfully assembled the effective iso-radius for the dynamic iso-1-sphere IHR [25] "base case" in the IHR topology [22, 31, 32, 33] and introduced the corresponding notion of effective iso-state to begin encoding the iso-transition between two distinct iso-states. For this, the procedure and step-by-step developing results were presented in Section 2, which apply to both continuous and discrete dynamic iso-sphere IHRs. Also, we demonstrated that all of these outcomes comply with the exterior and interior IHR inverse iso-duality [25]. To recapitulate the final results more precisely, we defined—for the dynamic iso-sphere IHR $T_{\hat{r}(m)}^1$ —the effective iso-radius $\hat{r}_E(m_0, m_1)$ from the *average* of $T_{\hat{r}(m_0)}^1$'s initial quantity m_0 and $T_{\hat{r}(m_1)}^1$'s final quantity m_1 , which correspond to the initial dynamic iso-sphere IHR $T_{\hat{r}(m_0)}^1$ and the final dynamic iso-sphere IHR $T_{\hat{r}(m_1)}^1$, respectively. Ultimately, we deployed these constructs to propose the new effective dynamic iso-1-sphere IHR $T_{\hat{r}_E(m_0, m_1)}$ to characterize the iso-transition $T_{\hat{r}(m_0)}^1 \rightarrow T_{\hat{r}(m_1)}^1$, which is a new iso-geometrical, iso-topological class of dynamic iso-sphere IHRs for a developing, generalized encoding approach.

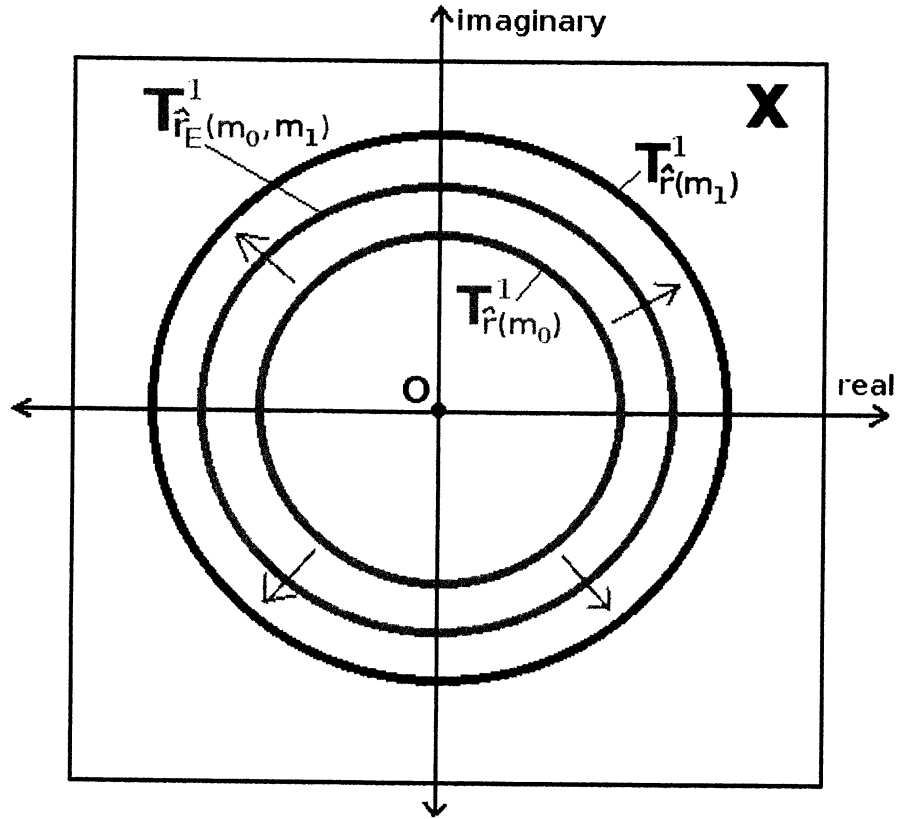


Fig. 5: For the iso-transition $T_{\hat{r}(m_0)}^1 \rightarrow T_{\hat{r}(m_1)}^1$ of Δ_m , $T_{\hat{r}(m_0)}^1$ is the initial dynamic iso-1-sphere IHR iso-state, $T_{\hat{r}(m_1)}^1$ is the final dynamic iso-1-sphere IHR iso-state, and $T_{\hat{r}_E(m_0, m_1)}^1$ is the characteristic effective dynamic iso-1-sphere IHR iso-state with the effective iso-radius $\hat{r}_E(m_0, m_1)$ and the effective iso-curvature $\hat{\kappa}_E(m_0, m_1)$.

The results, constructions, and implications of this preliminary investigation are significant because they exemplify alternative modes of cutting-edge iso-mathematics research that facilitate a heightened quantifiable characterization of dynamic iso-sphere IHRs [25] in terms of effective iso-states for iso-transitions with iso-duality. Hence, given that iso-sphere IHRs are equipped with topological deformation order parameters [32, 33, 22], a next logical step of this analysis could be to implement iso-topic liftings [1, 2, 3, 4, 5] for the order parameters and then topologically incorporate these “iso-deformations” into the existing effective iso-state definition. From there, we may build on this platform and continue to develop the framework by exploring and assessing the frontiers of iso-, geno-, and hyper-mathematics [1, 2, 3, 4, 5]. One of the questions that comes to mind is this: *do effective dynamic iso-sphere IHRs equipped with effective iso-radii have direct application to chemistry and/or physics such as, for example, the non-strictly thermal, non-strictly continuous energy spectrum of black holes [26, 27, 28, 29, 30] or hadronic mechanics [6, 7]?* Our hypothesis is: *yes*. Hence, in order to test this conjecture via the scientific method, this developing class of effective dynamic iso-sphere IHRs must be subjected to further development, scrutiny, collaboration, and hard work in order to advance it for future application in the disciplines of iso-mathematics, physics, and science in general.

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**CR-SUBMANIFOLDS OF A LORENTZIAN
PARA-SASAKIAN MANIFOLD ENDOWED WITH A
SEMI-SYMMETRIC NON-METRIC CONNECTION**

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Abstract

We define a semi-symmetric non-metric connection in a Lorentzian Para-Sasakian manifold and study CR-Submanifolds of a Lorentzian Para- Sasakian manifold endowed with a semi-symmetric non-metric connection. Moreover, We also obtain integrability conditions of the distributions on CR-Submanifolds.

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1. Introduction

The notion of CR-submanifolds of a Kaehler manifold was introduced by A. Bejancu. in [2]. Later, CR-submanifolds of Saskian manifold were studied by M. Kobayashi in [7]. K. Motsumoto introduced the idea of Lorentzian Para-Sasakian structure and studied its several properties in [8]. B. Prasad, [9] and S. Prasad, and R.H. Ojha, [10] studied submanifolds of a Lorentzian Para-Sasakian manifolds. U. C. De and Anup Kumar Sengupta studied CR-Submanifolds of a Lorentzian Para-Sasakian manifold in [4]. In this paper we study CR-Submanifolds of a Lorentzian Para-Sasakian manifold endowed with a semi-symmetric non-metric connection.

Let ∇ be a linear connection in an n -dimensional differentiable manifold $\bar{M}$. The Torsion tensor T and the Curvature tensor R of ∇ are given respectively by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

The connection ∇ is symmetric if torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is metric connection if there is a Riemannian metric g in $\bar{M}$ such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection.

In ([5], [11]) A. Friedmann, and J.A. Schouten, introduced the idea of a semi-symmetric linear connection. A linear connection ∇ is said to be semi-symmetric connection if its torsion tensor T is of the form

$$T(X, Y) = \eta(Y)X - \eta(X)Y,$$

where η is a 1-form. In [12], K. Yano studied some properties of semi-symmetric metric connection. In [1] N.S. Agashe and M.R. Chaffle studied some properties of semi-symmetric non-metric connection. In [8], Mobin Ahmad and C. Ozgur defined a semi-symmetric non-metric connection and studied properties of hypersurfaces of almost r -paracontact Riemannian manifold with semi-symmetric non-metric connection.

In this paper we study CR-Submanifolds of a Lorentzian Para-Sasakian manifold endowed with a semi-symmetric non-metric connection. We consider integrabilities of horizontal and vertical distributions of CR-submanifolds with a semi-symmetric non-metric connection. We also consider parallel horizontal distributions of CR-submanifolds.

The Paper is organized as follows : In section 2, we give a brief introduction of Lorentzian para-Sasakian manifold. In section 3, we show that the induced connection on CR-submanifolds of LP-Sasakian manifold with semi-symmetric non-metric connection is also a semi-symmetric non-metric connection. In section 4, we established some lemmas on Integrabilities of horizontal and vertical distributions and in the last section we

discussed the integrability conditions of Parallel horizontal distributions of CR-submanifolds.

2. Preliminaries

Let $\bar{M}$ be an n -dimensional almost contact metric manifold with almost contact metric structure (ϕ, η, ξ, g) . Then we have by definition

$$(2.1) \quad \phi^2 X = X + \eta(X)\xi, \eta(\xi) = -1$$

$$(2.2) \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

$$(2.3) \quad g(X, \xi) = \eta(X)$$

$$(2.4) \quad g(\phi X, Y) = g(X, \phi Y) = \phi(X, Y)$$

for all vector fields X, Y tangent to $\bar{M}$. Such a structure (ϕ, η, ξ, g) is termed as Lorentzian para-contact structure [8].

Also in a Lorentzian para-contact structure the following relations hold:

$$\phi\xi = 0, \eta(\phi X) = 0, \text{rank}(\phi) = n - 1.$$

A Lorentzian para-contact manifold $\bar{M}$ is called Lorentzian para-Sasakian (LP-Sasakian manifold if [8]).

$$(2.5) \quad (\bar{\nabla}_X \phi)(Y) = g(X, Y) + \eta(Y)X + 2\eta(X)\eta(Y)\xi.$$

$$(2.6) \quad \bar{\nabla}_X \xi = \phi X$$

for all vector fields X, Y tangent to $\bar{M}$, where $\bar{\nabla}$ is the Riemannian connection with respect to g .

Definition : An m -dimensional Riemannian submanifold M of a Lorentzian para-Sasakian manifold $\bar{M}$ is called a CR-submanifold if ξ is tangent to M and there exists on M a differentiable distribution $D : x \rightarrow D_x \subset T_x(M)$ such that

- (i) D_x is invariant under ϕ i.e. $\phi D_x \subset D_x$ for each $x \in M$.
- (ii) The orthogonal complementary distribution $D^\perp : x \rightarrow D_x^\perp \subset T_x(M)$ of the distribution D on M is totally real i.e. $\phi D_x^\perp(M) \subset T_x^\perp(M)$ where $T_x(M)$ and $T_x^\perp(M)$ are tangent space and normal space of M at $x \in M$.

The distribution D (resp. $D^\perp$) is called the horizontal (resp. vertical distribution). The pair $(D, D^\perp)$ is called ξ -horizontal (resp. ξ -vertical) if $\xi_x \in D_x$ (resp. $\xi_x \in D_x^\perp$) for $x \in M$.

Let us denote the orthogonal complement of $\phi D^\perp$ in $T^\perp(M)$ by μ . Then we have,

$$TM = D \oplus D^\perp, T^\perp(M) = \phi D^\perp \oplus \mu.$$

It is obvious that $\phi\mu = \mu$.

Now, we define Semi-symmetric metric connection $\bar{\nabla}$ as

$$(2.7) \quad \bar{\nabla}_X Y = \bar{\nabla}_X Y + \eta(Y)X$$

such that $\bar{\nabla}_X g = 0$ for any $X, Y \in TM$ where $\bar{\nabla}$ is induced connection with respect to g on M .

Inserting (2.7) in (2.4), we have

(2.8)(a)

$$(\bar{\nabla}_X \phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi - \eta(Y)\phi X,$$

(2.8)(b)

$$\bar{\nabla}_X \xi = \phi X + X$$

We denote by g the metric tensor of $\bar{M}$ as well as that induced on M . Let $\bar{\nabla}$ be the semi-symmetric metric connection on $\bar{M}$ and ∇ be the induced connection on M with respect to unit normal N . Then

Theorem 2.1. The connection induced on CR-submanifolds of LP-Sasakian manifold with semi-symmetric non-metric connection is also a semi-symmetric non-metric connection.

Proof : Let ∇ be the induced connection with respect to unit normal N on CR-submanifolds of LP-Sasakian manifold from semi-symmetric metric connection $\bar{\nabla}$, then

(2.9)

$$\bar{\nabla}_X Y = \nabla_X Y + m(X, Y),$$

where m is a tensor field of type $(0, 2)$ on CR-submanifolds M . If $\dot{\nabla}$ be the induced connection on CR-submanifolds from Riemannian connection $\bar{\bar{\nabla}}$, then

(2.10)

$$\bar{\bar{\nabla}}_X Y = \dot{\nabla}_X Y + h(X, Y),$$

where h is a second fundamental tensor satisfying

$$h(\bar{X}, \bar{Y}) = h(\bar{Y}, \bar{X}) = g(H(\bar{X}), \bar{Y})$$

By the definition of semi-symmetric non-metric connection

$$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y + \eta(Y)X$$

Now, using above equations, we have

$$\nabla_X Y + m(X, Y) = \bar{\nabla}_X Y + h(X, Y) + \eta(Y)X$$

Equating tangential and normal components from both the sides, we get

$$h(X, Y) = m(X, Y)$$

and $\nabla_X Y = \bar{\nabla}_X Y + \eta(Y)X$.

Thus ∇ is also a semi-symmetric non-metric connection for any $X, Y, \xi \in TM$.

Now, Gauss equation for CR-submanifolds of LP-Sasakian manifold with semi-symmetric non-metric connection is

$$(2.11) \quad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

and Weingarten formula is given by

$$(2.12) \quad \bar{\nabla}_X N = -A_N X + \nabla_X^\perp N + \eta(N)X$$

for $X, Y \in TM$, $N \in T^\perp M$, h (resp. A_N) is the second fundamental form (resp. tensor) of M in $\bar{M}$ and $\nabla^\perp$ denotes the operator of the normal connection. Moreover, we have

$$(2.13) \quad g(h(X, Y), N) = g(A_N X, Y).$$

Any vector X tangent to M is given as

$$(2.14) \quad X = PX + QX,$$

where PX and QX belong to the distribution D and $D^\perp$ respectively.

For any vector field N normal to M , we put

$$(2.15) \quad \phi N = BN + CN$$

where BN (resp. CN) denotes the tangential (resp. normal) component of ϕN .

3. Integrabilities of horizontal distribution D and vertical distribution $D^\perp$

Lemma 3.1. Let M be a CR-submanifold of an LP-Sasakian manifold $\bar{M}$ with semi-symmetric non-metric connection, then

$$(3.1) \quad \begin{aligned} P\nabla_X \phi PY - PA_{\phi QY} X &= \phi P\nabla_X Y + g(X, Y)P\xi + \eta(Y)PX \\ &\quad - \eta(Y)\phi PX + 2\eta(X)\eta(Y)P\xi \end{aligned}$$

$$(3.2) \quad \begin{aligned} Q\nabla_X \phi PY - QA_{\phi QY} X &= g(X, Y)Q\xi + \eta(Y)QX \\ &\quad - \eta(Y)\phi QX + 2\eta(X)\eta(Y)Q\xi + Bh(X, Y) \end{aligned}$$

$$(3.3) \quad h(X, \phi PY) + \nabla_X^\perp \phi QY = \phi Q\nabla_X Y + Ch(X, Y)$$

for $X, Y \in TM$.

Proof : By virtue of (2.8)(a), (2.11), (2.12), (2.14) and (2.15), we can easily get

$$P\nabla_X \phi PY + Q\nabla_X \phi PY + h(X, \phi PY) - PA_{\phi QY} X - QA_{\phi QY} X +$$

$$\begin{aligned}
 +\nabla_X^\perp \phi QY &= g(X, Y)P\xi + g(X, Y)Q\xi + \eta(Y)PX + \eta(Y)QX - \\
 &\quad -\eta(Y)\phi PX - \eta(Y)\phi QX + 2\eta(X)\eta(Y)P\xi + 2\eta(X)\eta(Y)Q\xi \\
 &\quad + \phi P\nabla_X Y + \phi Q\nabla_X Y + Bh(X, Y) + Ch(X, Y)
 \end{aligned}$$

Equations (3.1)-(3.3) follows by equating horizontal, vertical and normal components.

Lemma 3.2. Let M be a CR-submanifold of an LP-Sasakian manifold $\bar{M}$ with semi-symmetric non-metric connection, then

$$\phi P[X, Y] = A_{\phi Y}Z - A_{\phi Z}Y + \eta(Y)Z - \eta(Z)Y - \eta(Y)\phi Z + \eta(Z)\phi Y$$

for $Y, Z \in D^\perp$.

Proof : By virtue of (2.8)(a), (2.11) and (2.12) we have for $Y, Z \in D^\perp$

$$\begin{aligned}
 -A_{\phi Z}Y + \nabla_Y^\perp \phi Z &= g(Y, Z)\xi + \eta(Z)Y - \eta(Z)\phi Y + \\
 &\quad + 2\eta(Y)\eta(Z)\xi + \phi(\nabla_Y Z + h(Y, Z)).
 \end{aligned}$$

Using (3.3), we get

$$\begin{aligned}
 \phi P\nabla_Y Z &= -A_{\phi Z}Y - g(Y, Z)\xi - \eta(Z)Y + \\
 &\quad + \eta(Z)\phi Y - 2\eta(Y)\eta(Z)\xi - Bh(Y, Z).
 \end{aligned}$$

Interchanging Y and Z , we have

$$\begin{aligned}
 \phi P\nabla_Z Y &= -A_{\phi Y}Z - g(Z, Y)\xi - \eta(Y)Z + \eta(Y)\phi Z - \\
 &\quad - 2\eta(Y)\eta(Z)\xi - Bh(Z, Y).
 \end{aligned}$$

On subtracting above two equations, we have

$$\phi P[Y, Z] = A_{\phi Y}Z - A_{\phi Z}Y + \eta(Y)Z - \eta(Z)Y + \eta(Z)\phi Y - \eta(Y)\phi Z$$

for $Y, Z \in D^\perp$. Thus we have

Theorem 3.3. Let M be a CR-submanifold of an LP-Sasakian manifold $\bar{M}$ with semi-symmetric non-metric connection. Then the distribution $D^\perp$ is integrable if and only if

$$A_{\phi Z}Y - A_{\phi Y}Z = \eta(Y)Z - \eta(Z)Y + \eta(Z)\phi Y - \eta(Y)\phi Z$$

for $Y, Z \in D^\perp$.

Proposition 3.4. Let M be a ξ -vertical CR-submanifold of an Lorentzian Para Sasakian manifold $\bar{M}$. Then

$$(3.4) \quad \phi Ch(X, Y) = Ch(\phi X, Y) = Ch(X, \phi Y)$$

for $X, Y \in D$.

Proof. From (3.2), we have for $X, Y \in D$

$$(3.5) \quad Q\nabla_X \phi Y = g(X, Y)Q\xi + 2\eta(X)\eta(Y)Q\xi + Bh(X, Y).$$

Replacing X by ϕX , we have

$$(3.6) \quad Q\nabla_{\phi X} \phi Y = g(\phi X, Y)Q\xi + Bh(\phi X, Y).$$

Using (3.5) and (2.1), we get

$$(3.7) \quad Q\nabla_X Y = g(\phi X, Y)Q\xi + Bh(\phi X, Y).$$

Adding (3.6) and (3.7), we have

$$(Q\nabla_X Y + Q\nabla_{\phi X} \phi Y) \in D$$

Replacing X by ϕX and Y by ϕY in (3.3), we have

$$(3.8) \quad -h(\phi X, Y) = \phi Q(\nabla_{\phi X} \phi Y) + Ch(\phi X, \phi Y).$$

Also interchanging X and Y in (3.3), we have

$$(3.9) \quad h(\phi X, Y) = \phi Q(\nabla_X Y) + Ch(X, Y).$$

Adding (3.8) (3.9) and (2.1) and using $(Q\nabla_X Y + Q\nabla_{\phi X} \phi Y) \in D$, we get

$$Ch(\phi X, Y) = Ch(X, \phi Y)$$

. From (3.5) and (2.1), we have

$$Q\nabla_X Y = g(X, \phi Y)Q\xi + 2\eta(X)\eta(Y)Q\xi + Bh(X, \phi Y).$$

Using above equation in (3.3), we get

$$Ch(\phi X, Y) = \phi Ch(X, Y)$$

which completes the theorem.

Theorem 3.5. Let M be a CR-submanifold of an LP-Sasakian manifold $\bar{M}$ with semi-symmetric non-metric connection. Then the distribution D is integrable if and only if $h(X, \phi Y) = h(Y, \phi X)$ for $Y, Z \in D$.

Proof : The proof of theorem is similar to theorem 3.2 in [4].

4. Parallel horizontal distributions of CR-submanifolds

Definition : The horizontal distribution D is said to be parallel with respect to the connection ∇ on M if $\nabla_X Y \in D$ for all vector fields $X, Y \in D$.

Proposition 4.1 Let M be a ξ -horizontal CR- submanifold of an LP-Sasakian manifold $\bar{M}$ with semi-symmetric non-metric connection. Then the distribution D is parallel if and only if

$$(4.1) \quad h(X, \phi Y) = h(Y, \phi X) = \phi h(X, Y)$$

for all $X, Y \in D$.

Proof : By similar computations as proposition 4.1 in [4], proposition follows.

Proposition(4.2): Let M be a ξ -vertical CR-submanifold of a Lorentzian Para-Sasakian manifold $\bar{M}$ with semi-symmetric non-metric connection. Then the distribution $D^\perp$ is parallel with respect to the connection ∇ on M , if and only if, $A_N X \in D^\perp$ for each $X \in D^\perp$ and $N \in TM^\perp$.

Proof: Let $Y, X \in D^\perp$. Then using (2.11) and (2.12), we have

$$\begin{aligned} -A_{\phi Y} X + \nabla_X^\perp \phi Y &= \phi \nabla_X Y + \phi h(Y, X) + \\ &+ \eta(Y)X + g(Y, X)\xi - \eta(Y)\phi X + 2\eta(X)\eta(Y)\xi. \end{aligned}$$

Taking inner product with $Z \in D$, we have

$$g(A_{\phi Y} X, Z) = g(\nabla_X Y, \phi Z)$$

Therefore, $\nabla_X Y = 0$ if and only if $A_{\phi Y} X \in D^\perp$ for all $X \in D^\perp$. From which our assertion follows.

Definition : A CR-submanifold M of an LP-Sasakian manifold $\bar{M}$ with semi-symmetric non-metric connection is said to be totally geodesic if $h(X, Y) = 0$. for $X \in D$ and $Y \in D^\perp$.

It follows immediately that a CR-submanifold is mixed totally geodesic if and only if $A_N X \in D$ for each $X \in D$ and $N \in T^\perp M$.

Let $X \in D$ and $Y \in \phi D^\perp$. For a mixed totally geodesic ξ -vertical CR-submanifold M of an LP-Sasakian manifold $\bar{M}$

with semi-symmetric non-metric connection, then from (2.8)(a) we have

$$(\bar{\nabla}_X \phi)N = 0$$

Since $\bar{\nabla}_X \phi N = (\bar{\nabla}_X \phi)N + \phi(\bar{\nabla}_X N)$ so that $\bar{\nabla}_X \phi N = \phi(\bar{\nabla}_X N)$. Using (2.11) and (2.12) in above equation, we have

$$\nabla_X(\phi N) = -\phi A_N X + \phi \nabla_X^\perp N$$

as $\phi A_N X \in D$, so that $\nabla_X \phi N \in D$ if and only if $\phi \nabla_X^\perp N = 0$. Thus we have the following proposition.

Theorem 4.2 Let M be a mixed totally ξ -vertical CR- submanifold of an LP-Sasakian manifold $\bar{M}$ with semi-symmetric non-metric connection. Then the normal section $N \in \phi D^\perp$ is D - parallel if and only if $\nabla_X(\phi N) \in D$ for $X \in D$.

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**ON THE PROPERTIES OF THE REGULAR DIGRAPH
ASSOCIATED TO A POSET**

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Abstract

Let $(P, \leq)$ be a poset with the least element 0. The undirected regular graph of ideals of P , denoted by $\Gamma(P)$, is a graph with all nontrivial ideals of P being the vertex-set and, for two distinct vertices I and J , I is adjacent to J if and only if I contains a non zero-divisor on J or J contains a non zero-divisor on I . In this paper, we study some properties of $\Gamma(P)$. For example, we completely determine when a regular graph is complete bipartite and split. Also we study some results related to the regular graphs of atomic posets.

AMS Classification 05C10, 06A07

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1 Introduction

There are many papers which interlink graph theory and poset theory. Several classes of graphs associated with algebraic structures have been actively investigated. For example, Cayley graphs have been studied in [11] and [12], zero-divisor graphs have been studied in [5], [6], [7], [10], and cozero-divisor graphs have been introduced and investigated in [2], [3] and [4].

Also, method of trees was invented to elucidate some problems of Lie group representations too. It was developed in 1960-1970-th by Vilenkin-Kuznetsov-Smorodinsky (VKS) [13]. The further development of Vilenkin-Kuznetsov-Smorodinsky method to Vilenkin-Kuznetsov-Moskaliuk-Smorodinsky-Smirnov-(VKMSS)-Trees was given by S.S. Moskaliuk and Yu.F. Smirnov [14, 16 – 20] on the base of conception of complementarity of Lie groups in sense of Moshinsky which exists between groups $O(n)$ and $Sp(2, R)$.

Recently, in [21], Nikmehr and Shaveisi defined the regular digraph of ideals of R , denoted by $\overrightarrow{\Gamma}_{\text{reg}}(R)$, which is a digraph whose vertex-set is the set of all non-trivial ideals of R and, for every two distinct vertices I and J , there is an arc from I to J , denoted by $I \rightarrow J$, whenever $I \not\subseteq Z(J)$. In other words, I contains a J -regular element. They studied and investigated some properties of this graph.

Let $(P, \leq)$ be a partially ordered set (poset, briefly) with the least element 0 and I be a non-empty subset of P . We say that I is an ideal of P if, for arbitrary elements x and y in P , the relations $x \in I$ and $y \leq x$ imply that $y \in I$. Also, we put

$$Z(I) = \{x \in P \mid \text{there exists a nonzero element } y \in I \\ \text{such that } \{x, y\}^l = \{0\}\},$$

where $\{x, y\}^l$ is the set of all elements t such that $t \leq x$ and $t \leq y$. Let $I(P)$ be the set of all ideals of P . An element $r \in L$ is called I -regular if $r \notin Z(I)$, where $I \in I(P)$.

The regular digraph of ideals of P , denoted by $\overrightarrow{\Gamma}(P)$, which is defined and studied in [1], is a digraph whose vertex-set is the set of all nontrivial ideals of P and, for every two distinct vertices I and J , there is an arc from

I to J whenever I contains a J -regular element, or equivalently $I \not\subseteq Z(J)$. Also, the undirected regular graph of ideals of P , denoted by $\Gamma(P)$, is a graph with all nontrivial ideals of P being the vertex-set, and, for two distinct vertices I and J , I is adjacent to J if and only if $I \not\subseteq Z(J)$ or $J \not\subseteq Z(I)$.

In [1], the authors studied some basic properties of $\vec{\Gamma}(P)$, and also they investigated the planarity of $\Gamma(P)$.

In this paper we continue studying the properties of the regular graph of ideals associated to a poset. For example, we completely determine when a regular graph is complete bipartite and split. We also study some results related to the regular graphs associated to atomic posets.

Now, we recall some definitions and notations on graphs and partially ordered sets. We use the standard terminology of graphs in [8] and partially ordered sets in [9].

A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if it is connected with diameter one. We use K_n to denote the complete graph with n vertices. An *independent set* of G is a subset of the vertices of G such that no two vertices in the subset represent an edge of G . The *independence number* of G , denoted by $\alpha(G)$, is the cardinality of the largest independent set. For a positive integer r , an *r -partite graph* is one whose vertex set can be partitioned into r subsets, so that no edges has both ends in any one subset. A *complete r -partite graph* is one in which each vertex is joined to every vertex that is not in the same part.

In a partially ordered set $(P, \leq)$ with a least element 0 , an element a is called an *atom* if $a \neq 0$, and, for an element x in P , the relation $0 \leq x \leq a$ implies either $x = 0$ or $x = a$. Assume that S is a subset of P . Then an element x in P is a *lower bound* of S if $x \leq s$ for all $s \in S$. An *upper bound* is defined in a dual manner. The set of all lower bounds of S is denoted by S^ℓ and the set of all upper bounds of S by S^u , that is,

$$S^\ell := \{x \in P \mid x \leq s, \text{ for all } s \in S\}$$

and

$$S^u := \{x \in P \mid s \leq x, \text{ for all } s \in S\}.$$

Now, suppose that I is a non-empty subset of P . A poset P is *atomic*, if every non-zero element is an atom. In this paper, we assume that P is a finite poset and $A(P) = \{a_1, a_2, \dots, a_n\}$ is the set of all atoms of P .

2 Basic definitions and properties

We begin this section with the following lemma.

Lemma 2.1. *Let P be a poset. Then if $\Gamma(P)$ is a complete bipartite graph, then $|A(P)| \leq 2$.*

Proof. Assume that X and Y are two parts of $\Gamma(P)$. Suppose on the contrary that $|A(P)| \geq 3$. Then $\{0, a_1\}$, $\{0, a_2\}$ and $\{0, a_3\}$ are in the same part, say part X , where $a_1, a_2, a_3 \in A(P)$. Now, if $\{0, a_1, a_2\}$ is in part X , then it is adjacent to $\{0, a_1\}$ and $\{0, a_2\}$, which is impossible. So we have that the vertex $\{0, a_1, a_2\}$ is in part Y . But then $\{0, a_3\}$ is not adjacent to $\{0, a_1, a_2\}$ which is a contradiction. $\square$

Clearly, if $|A(P)| = 1$, then $\Gamma(P)$ is a complete graph. Thus we have the following corollary:

Corollary 2.2. *Let P be a poset and $|A(P)| = 1$. Then the following conditions are equivalent.*

- (i) $\Gamma(P)$ is a complete bipartite graph.
- (ii) P is a chain with $|P| \leq 4$.
- (iii) $\Gamma(P)$ is a star graph.

Proposition 2.3. *Let P be a poset. Then $\Gamma(P)$ is complete bipartite if and only if P is a chain with $|P| \leq 4$, or P is one of the posets in Figure 1.*

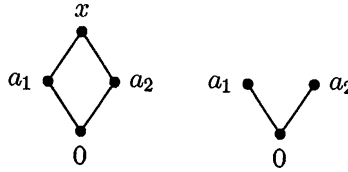


Figure 1

Proof. First suppose that $\Gamma(P)$ is a complete bipartite graph and P is non of the posets in Figure 1. By Lemma 2.1, we have $|A(P)| \leq 2$. By Corollary 2.2, we have $|A(P)| = 2$. Set $A(P) = \{a_1, a_2\}$ and suppose that X and Y are two parts of $\Gamma(P)$. Now, we have the following two cases:

Case 1. There exists an element $x \in \{a_i\}^u \setminus \{a_j\}^u$, where $i \neq j$ and $i, j \in \{1, 2\}$. Without loss of generality, we may assume that $x \in \{a_1\}^u \setminus \{a_2\}^u$. Clearly $\{0, a_1\}$ and $\{0, a_2\}$ belongs to a same part, say X . Then $\{0, a_1, x\}$ and $\{0, a_1, a_2\}$ belongs to Y . But this is impossible, since $\{0, a_1, x\}$ is adjacent to $\{0, a_1, a_2\}$.

Case 2. Suppose that for all $x \in P \setminus \{0, a_1, a_2\}$, we have $x \in \{a_1, a_2\}^u$. In this case the vertices $\{0, a_1\}$ and $\{0, a_2\}$ belong to a same part, say X . Also $\{0, a_1, a_2, x\}$ and $\{0, a_1, a_2\}$ belong to Y , which is again impossible.

Therefore, if $\Gamma(P)$ is a complete bipartite graph, then P is a chain with $|P| \leq 4$, or it is one of the posets in Figure 1.

The converse statement is clear. □

A graph $\Gamma(P)$ is said to be a *split* graph if the vertex-set of $\Gamma(P)$ is a disjoint union of two sets K and S , where K induces a complete subgraph and S is an independent set.

Lemma 2.4. *Let P be a poset such that $\Gamma(P)$ is split. Then $|A(P)| \leq 2$.*

Proof. Let K and S be two parts of $\Gamma(P)$ such that K is complete and S is independent. Assume on the contrary that $|A(P)| \geq 3$. Suppose that $a_1, a_2, a_3 \in A(P)$. If $\{0, a_1\} \in K$, then $\{0, a_2\}$ must be in S , because $\{0, a_1\}$ is not adjacent to $\{0, a_2\}$. Now, we have the following two cases:

Case 1. If $\{0, a_3\}$ is in K , then K is not complete, because $\{0, a_1\}$ is not adjacent to $\{0, a_3\}$.

Case 2. If $\{0, a_3\}$ is in S , then $\{0, a_2, a_3\}$ must be in K , which is a contradiction, because $\{0, a_2, a_3\}$ is not adjacent to $\{0, a_1\}$.

So if $\Gamma(P)$ is split, then $|A(P)| \leq 2$. □

Proposition 2.5. *Let P be a poset. Then $\Gamma(P)$ is split if and only if one of the following conditions holds.*

$$(i) |A(P)| = 1.$$

$$(ii) |A(P)| = 2 \text{ and } |P| \leq 4.$$

$$(iii) \{a_i\}^u \setminus \{a_j\}^u = \emptyset, \text{ for all } 1 \leq i \neq j \leq 2.$$

Proof. First suppose that $\Gamma(P)$ is split and $|A(P)| \neq 1$. Then, by Lemma 2.4, we have $|A(P)| = 2$. Now, suppose on the contrary that $|P| \geq 5$ and $\{a_i\}^u \neq \emptyset$, for some $1 \leq i \leq 2$. Then we have the following situations:

There exist distinct elements $x_1, x_2 \in P \setminus \{0, a_1, a_2\}$ such that $x_1, x_2 \in \{a_i\}^u \setminus \{a_j\}^u$, or $x_1 \in \{a_i\}^u \setminus \{a_j\}^u$ and $x_2 \in \{a_1, a_2\}^u$, or $x_1 \in \{a_i\}^u \setminus \{a_j\}^u$ and $x_2 \in \{a_j\}^u \setminus \{a_i\}^u$, for some $1 \leq i \neq j \leq 2$ and $a_1, a_2 \in A(P)$.

So we have that

$$\{\{0, a_1\}, \{0, a_1, x_1\}, \{0, a_1, x_1, x_2\}, \{0, a_1, a_2\}\} \subseteq K \text{ and } \{\{0, a_2\}\} \subseteq S.$$

Now consider the vertex $u := \{0, a_1, a_2, x_1\}$. If $u \in K$, then u is not adjacent to $\{0, a_1, a_2\}$. So u is in S . But it is impossible, because u is adjacent to $\{0, a_2\}$, and so $\Gamma(P)$ is not split. Therefore, $|A(P)| = 1$, or $|A(P)| = 2$ and $|P| \leq 4$ or $\{a_i\}^u \setminus \{a_j\}^u = \emptyset$, for all $1 \leq i \neq j \leq 2$.

Conversely, if $|A(P)| = 1$, then it is easy to see that $\Gamma(P)$ is split.

Now, if $|A(P)| = 2$, $|P| \leq 4$ or $\{a_i\}^u \setminus \{a_j\}^u = \emptyset$, for all $1 \leq i \neq j \leq 2$, then we have the following two cases:

Case 1. If there exists an element $x \in \{a_i\}^u \setminus \{a_j\}^u$, for some $1 \leq i \neq j \leq 2$, then

$$K = \{\{0, a_1\}, \{0, a_1, a_2\}, \{0, a_1, x\}\} \text{ and } S = \{\{0, a_2\}\},$$

and so $\Gamma(P)$ is split.

Case 2. If $\{a_i\}^u \setminus \{a_j\}^u = \emptyset$, for all $1 \leq i \neq j \leq 2$, then

$$K = V(\Gamma(P)) \setminus \{0, a_2\} \text{ and } S = \{\{0, a_2\}\},$$

which means that $\Gamma(P)$ is split.

Now, suppose that $|P| \leq 4$ and $\{a_i\}^u \setminus \{a_j\}^u = \emptyset$, for all $1 \leq i \neq j \leq 2$. If $|P| = 3$, then clearly $\Gamma(P)$ is split. If $|P| = 4$, then

$$K = V(\Gamma(P)) \setminus \{0, a_2\} \text{ and } S = \{\{0, a_2\}\},$$

and so $\Gamma(P)$ is split. $\square$

In a graph G , a vertex v is called *simplicial* if the subgraph of G induced by the vertex set $\{v\} \cup N(v)$ is a complete graph.

Proposition 2.6. *Let P be a poset. Then if P is a chain, then all vertices in $\Gamma(P)$ is a simplicial vertex.*

Proof. Suppose that P is a chain. Then the graph $\Gamma(P)$ is a complete graph, and so all vertices are simplicial. $\square$

Proposition 2.7. *Let P be a poset. Then if I is a simplicial vertex, then I has only one atom.*

Proof. Suppose on the contrary that I has at least two atoms. Set $A(P) = \{a_1, a_2\}$. In this case, I is adjacent to $\{0, a_1\}$ and $\{0, a_2\}$, but $\{0, a_1\}$ is not adjacent to $\{0, a_2\}$, a contradiction. So I has only one atom. $\square$

The following example shows that the converse of Lemma 2.7 is not true in general.

Example 2.8. *Suppose that P is a poset in Figure 2. Then it is easy to see that the vertex $I = \{0, a_2\}$, which has only one atom, is not a simplicial vertex.*

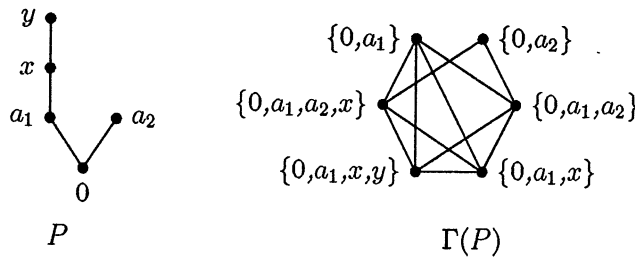


Figure 2

In the next theorem, we provide some conditions under which the converse of Lemma 2.7 holds.

Proposition 2.9. *Let P be a poset and $I \in V(\Gamma(P))$. If I has only one atom and P is one of the following posets in Figure 3, then I is a simplicial vertex.*

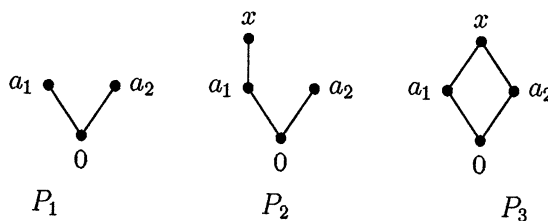


Figure 3

Proof. Assume that P is one of the posets in Figure 3, and $\Gamma(P_1)$, $\Gamma(P_2)$ and $\Gamma(P_3)$ are pictured in Figure 4.

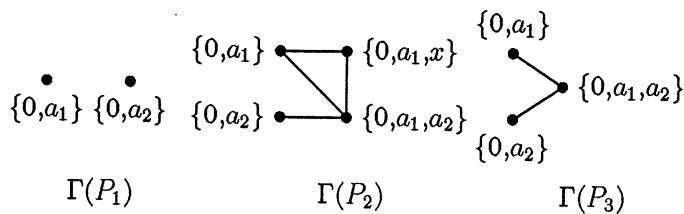


Figure 4

Now, one can easily see that all vertices with one atom in $\Gamma(P_1)$, $\Gamma(P_2)$ and $\Gamma(P_3)$ are simplicial vertices. $\square$

The *diameter* of a connected graph G is the maximum distance between two distinct vertices of G . For any vertex x of a connected graph G , the

eccentricity of x , denoted by $e(x)$, is the maximum of the distances from x to the other vertices of G . The set of vertices with minimal eccentricity is called the *center* of the graph, denoted by $C(G)$, and this minimum eccentricity value is the *radius* of G , denoted by $r(G)$. (We shall say the diameter and radius of G are zero if G has no edges.)

Notation 2.10. Let $i_1, i_2, \dots, i_n$ be integers with $1 \leq i_1 < i_2 < \dots < i_k \leq n$ and $A(P) = \{a_1, a_2, \dots, a_n\}$. The notation $U_{i_1 i_2 \dots i_k}$ stands for the following set:

$$\{I \trianglelefteq P; \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\} \subseteq I \text{ and } a_j \notin I, \text{ for } j \in \{1, \dots, n\} \setminus \{i_1, \dots, i_k\}\}.$$

Note that all two distinct elements in $U_{i_1 i_2 \dots i_k}$ are adjacent in $\Gamma(P)$. Also if the index sets $\{i_1, i_2, \dots, i_k\}$ and $\{j_1, j_2, \dots, j_{k'}\}$ of $U_{i_1 i_2 \dots i_k}$ and $U_{j_1 j_2 \dots j_{k'}}$, respectively, are distinct, then one can easily check that $U_{i_1 i_2 \dots i_k} \cap U_{j_1 j_2 \dots j_{k'}} = \emptyset$. Moreover, $V(\Gamma(P)) = \bigcup U_{i_1 i_2 \dots i_k}$, for all $1 \leq i_1 < i_2 < \dots < i_k \leq n$. Also by $1 \dots \hat{i} \dots n$ we mean that $1 \dots i - 1 \ i + 1 \dots n$.

Lemma 2.11. Let P be a poset with $|A(P)| = 3$. Then the following statements hold.

$$(i) \ r(\Gamma(P)) = 3; \text{ and,}$$

$$(ii) \ C(\Gamma(P)) = V(\Gamma(P)).$$

Proof. Suppose that I_1 and I_2 are two distinct vertices in $\Gamma(P)$. We have the following situations:

- (i) If $I_1, I_2 \in U_i$, for some $1 \leq i \leq 3$, then the vertex I_1 is adjacent to the vertex I_2 .
- (ii) If $I_1, I_2 \in U_{ij}$, for some $1 \leq i \neq j \leq 3$, then we have the path $I_1 - I_3 - I_2$, for some $I_3 \in I_i$ or $I_3 \in I_j$.
- (iii) If $I_1 \in U_{ij}$ and $I_2 \in U_{jk}$, for some $1 \leq i \neq j \neq k \leq 3$, then we have the path $I_1 - I_2 - I_3$, for some $I_3 \in I_j$.

- (iv) If $I_1 \in U_i$ and $I_2 \in U_{jk}$, for some $1 \leq i \neq j \neq k \leq 3$, then we have the path $I_1 - I_3 - I_4 - I_2$, for some $I_3 \in I_{ij}$ and $I_4 \in I_j$.

Now, by considering the above situations, we have that, for each $I \in V(\Gamma(P))$, $e(I) = 3$, and so $r(\Gamma(P)) = 3$ and $C(\Gamma(P)) = V(\Gamma(P))$. $\square$

In the sequel of this section, we investigate the eccentricity of the vertices and the center of the graph $\Gamma(P)$, when $|A(P)| \geq 4$. To this end, consider two distinct vertices I_1 and I_2 in $\Gamma(P)$. We have the following situations:

- (i) If $I_1, I_2 \in U_i$, for some $1 \leq i \leq n$, then the vertex I_1 is adjacent to the vertex I_2 .
- (ii) If $I_1, I_2 \in U_{i\dots j}$, for some $1 \leq i \neq j \leq n$, then we have the path $I_1 - I_3 - I_2$, for some $I_3 \in I_i$ or $I_3 \in I_j$.
- (iii) If $I_1 \in U_{i\dots j}$ and $I_2 \in U_{j\dots k}$, for some $1 \leq i \neq j \neq k \leq n$, then we have the path $I_1 - I_2 - I_3$, for some $I_3 \in I_j$.
- (iv) If $I_1 \in U_i$ and $I_2 \in U_{1\dots\hat{i}\dots j}$, for some $1 \leq i \neq j \leq n$, then we have the path $I_1 - I_3 - I_4 - I_2$, for some $I_3 \in I_{i\dots j}$ and $I_4 \in I_j$.
- (v) If $I_1 \in U_{i\dots j}$ and $I_2 \in U_{1\dots\hat{i}\dots\hat{j}\dots n}$, for some $1 \leq i \neq j \leq n$, then we have the path $I_1 - I_3 - I_4 - I_5 - I_2$, for some $I_3 \in U_i, I_4 \in U_{i\dots k}$ and $I_5 \in U_k$, for some $k \in 1\dots\hat{i}\dots\hat{j}\dots n$.

Now, from the above discussion, we have the following proposition.

Proposition 2.12. *Let P be a poset with $|A(P)| \geq 4$ and I be a vertex in $\Gamma(P)$. Then the following statements hold.*

- (i) *If $I \in U_i$, for some $1 \leq i \leq n$, then $e(I) = 3$. Otherwise, we have $e(I) = 4$.*
- (ii) $C(\Gamma(P)) = \bigcup_{i=1}^n U_i$.

3 The regular graphs associated to atomic posets

Throughout this section we assume that $P = \{0, a_1, \dots, a_n\}$ is an atomic poset, $n \geq 2$ and $a_i \in A(P)$ for all i with $1 \leq i \leq n$.

In this section, we study some basic properties of the regular graph of an atomic poset.

One can easily see that $\Gamma(P)$ is a bipartite graph with two parts X and Y , where $X = \{\{0, a_i\} \mid 1 \leq i \leq n\}$ and $Y = V(\Gamma(P)) \setminus X$.

Lemma 3.1. *In $\Gamma(P)$ we have,*

$$|V(\Gamma(P))| = \sum_{i=1}^{n-1} \binom{n}{i},$$

where $n = |A(P)|$.

Proof. We use induction on n . First suppose that $n = 2$ and set $A(P) = \{a_1, a_2\}$. Obviously, $\Gamma(P)$ has two vertices $\{0, a_1\}$ and $\{0, a_2\}$. Also we have

$$|V(\Gamma(P))| = \sum_{i=1}^1 \binom{2}{i} = \binom{2}{1} = 2.$$

Now, suppose that $n = k$ and

$$|V(\Gamma(P))| = \sum_{i=1}^{k-1} \binom{k}{i}.$$

If $n = k + 1$, then the sum of vertices of $\Gamma(P)$ are of the form

$$|V(\Gamma(P))| = \sum_{i=1}^{k-1} \binom{k}{i} + \sum_{i=1}^{k-1} \binom{k-1}{i-1} + \binom{k+1}{k}.$$

Therefore

$$|V(\Gamma(P))| = \sum_{i=1}^k \binom{k+1}{i}.$$

□

Proposition 3.2. *In $\Gamma(P)$ we have*

$$\deg(\{0, a_i\}) = 2^{n-1} - 2,$$

where $1 \leq i \leq n$.

Proof. By axiom of choice one can easily see that,

$$\deg(\{0, a_i\}) = \sum_{i=1}^{i=n-2} \binom{n-1}{i}.$$

Also, it is well-known that $\sum_{i=0}^{i=n-1} \binom{n}{k} = 2^{n-1}$. Therefore,

$$\deg(\{0, a_i\}) = 2^{n-1} - 2.$$

□

Let G and H be graphs. A *homomorphism* f from G to H is a map from $V(G)$ to $V(H)$ such that for any $a, b \in V(G)$, a is adjacent to b implies that $f(a)$ is adjacent to $f(b)$. Moreover, if f is bijective and its inverse mapping is also a homomorphism, then we call f an *isomorphism* from G to H , and in this case we say G is isomorphic to H , denoted by $G \cong H$. A homomorphism (resp, an isomorphism) from G to itself is called an *endomorphism* (resp, *automorphism*) of G . An endomorphism f is said to be *half-strong* if $f(a)$ is adjacent to $f(b)$ implies that there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that c is adjacent to d . By $\text{End}(G)$, we denote the set of all the endomorphisms of G . It is well-known that $\text{End}(G)$ is a monoid with respect to the composition of mappings. Let x be an arbitrary vertex of a graph G . The *neighborhood* (resp, *degree*) of x , denoted by $N(x)$ (resp, $\deg(x)$), is the set of vertices which are adjacent to x (resp, the cardinality of $N(x)$). Let S be a semigroup. An element a in S is called *regular* if $a = aba$ for some $b \in S$ and S is called *regular* if every element in S is regular. Also, a graph G is called *end-regular* if $\text{End}(G)$ is regular.

Lemma 3.3. *Let P be a poset, with $|A(P)| \geq 2$. Then, for two distinct vertices I and J in $V(\Gamma(P))$, $A(I) \subseteq A(J)$ if and only if $N(I) \subseteq N(J)$.*

Proof. First assume that $A(I) \subseteq A(J)$. Then, for all distinct vertices $k \in N(I)$ such that $k \in V(\Gamma(P)) \setminus I, J$, we have $k \in N(J)$, because $A(I) \subseteq A(J)$. So $N(I) \subseteq N(J)$.

Conversely, suppose that $N(I) \subseteq N(J)$. Then all the vertices that adjacent to I , are adjacent to J . So $A(I) \subseteq A(J)$. $\square$

Now, we investigate end-regularity of the graph $\Gamma(P)$. Clearly, if $|A(P)| \leq 2$, then $\Gamma(P)$ is end-regular.

Now, we recall the following Lemma from [15].

Lemma 3.4. [15, Lemma 2.1] *Let G be a graph. If there are pairwise distinct vertices a, b, c in G satisfying $N(c) \subsetneq N(a) \subseteq N(b)$, then G is not end-regular.*

Theorem 3.5. *Let P be a poset, with $|A(P)| \geq 5$. Then $\Gamma(P)$ is not end-regular.*

Proof. Suppose that $I_1, I_2, I_3 \in V(\Gamma(P))$ such that $I_1 \in U_{i\dots j}, I_2 \in U_{i\dots j\dots k}$ and $I_3 \in U_{i\dots j\dots k\dots t}$, for some $1 \leq i \neq j \neq k \neq t \leq n$. Then, by Lemma 3.3, $N(I_1) \subsetneq N(I_2)$. Now, we have $N(I_1) \subsetneq N(I_2) \subsetneq N(I_3)$, and so, by Lemma 3.4, $\Gamma(P)$ is not end-regular. $\square$

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SOME BOL-MOUFANG CHARACTERIZATION OF THE THOMAS PRECESSION OF A GYROGROUP^{1 2 3}

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Abstract

Strictly speaking, among all loops which are related to gyrogroups and gyrocommutative gyrgroups, only the left Bol loop is of Bol-Moufang type. In this study, the relationship between gyrogroups, gyrocommutative gyrgroups and other loops of Bol-Moufang types such as right alternative loop (RAL), flexible loop (FL), weak inverse property loop (WIPL), right central loop (RCL), left central loop (LCL), central loop (CL), right nuclear square loop (RNSL), left nuclear square loop (LNSL), middle nuclear square loop (MNSL), right Bol loop (RBL), Moufang loop (ML) and extra loop (EL) are investigated. In a gyrogroup $(G, \oplus)$ with Thomas gyration $gyr[x, y]$: RAL, WIPL and FL are equivalent and characterized by $gyr[x, y]x = x$, RNSL is characterized by $gyr[z, y](x \oplus x) = x \oplus x$, LCL, LNSL, MNSL are equivalent and characterized by $gyr[x \oplus x, y] = I$, EL, CL, RCL are equivalent and characterized by $gyr[x, y]L_x gyr[x, y] = L_x$, RBL and ML are equivalent and characterized by $gyr[x, y] = L_x gyr[x, x \oplus y]L_x^{-1}$. Some new properties of the Thomas Precession of some loops of Bol-Moufang type gyrogroup and gyrocommutative gyrgroup are also deduced.

¹Dedicated to Professor A.R.T. Solarin on his 60th Birthday Celebration.

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1 Introduction

Amongst mathematicians and physicists, symmetry is thought as being virtually synonymous with the theory of groups and their actions. However, unlike velocity addition in the non-relativistic setting, Einsteins velocity addition is not a group operation because it is not associative. It turns out, however, that owing to the presence of the Thomas precession there is no loss of mathematical regularity and symmetry associated with the breakdown of associativity. The mathematical regularity that the Thomas precession stores has been deciphered, and it extends it by abstraction to the so called Thomas gyration.

In the wide area of nonassociativity in physics and mathematics, the Einstein velocity addition is a special case, serving as a model of an abstract structure, called a gyrogroup, which involves nonassociativity and noncommutativity which are both 'repairable' by gyroautomorphisms called Thomas gyrations. This, in turn, gave birth to the "gyroterminology": the pre- fix "gyro" in terms like gyrogroups, gyrovectors spaces, gyrocommutativity and gyroassociativity laws to emphasize analogies shared with respective classical counterparts; groups, vector spaces, commutativity and associativity.

Relativistically admissible velocities are elements of the open ball $\mathbb{R}_c^3$ of radius c in the Euclidean three-space,

$$\mathbb{R}_c^3 = \{v \in \mathbb{R}^3 : \|v\| < c\} \quad (1)$$

c being a positive constant that represents the vacuum speed of light. The Einstein velocity addition is a binary operation $\oplus_E$ in the ball $\mathbb{R}_c^3$ given by the equation

$$u \oplus_E v = \frac{1}{1 + \frac{u \cdot v}{c^2}} \left\{ u + v + \frac{1}{c^2} \frac{\gamma_u}{1 + \gamma_u} (u \times (u \times v)) \right\} \quad (2)$$

for $u, v \in \mathbb{R}_c^3$. Here $\cdot$ and $\times$ represent the usual dot and cross products in $\mathbb{R}^3$, $\|\cdot\|$ is a norm on $\mathbb{R}^3$ and γ_u is the Lorentz factor given by $\gamma_u = \frac{1}{\sqrt{1 - \frac{\|u\|^2}{c^2}}}$.

When u and v are non-parallel, Einsteins addition (2) is neither commutative nor associative. In the special case when the velocity vectors are parallel, Einsteins velocity addition takes the simpler form

$$u \oplus v = \frac{u + v}{1 + \frac{u \cdot v}{c^2}} \quad (3)$$

for $u, v \in \mathbb{R}_c^3$ such that $u \parallel v$. In this case the operation $\oplus$ is both commutative and associative. Indeed, as Poincaré pointed out in 1906/1907, Einsteins addition in (3) of relativistically admissible parallel velocities is a group operation, thus admitting a useful algebraic structure. More than 80 years later it was discovered that also the general Einstein velocity addition is not algebraically structureless; rather, it is a gyrocommutative gyrogroup operation according to Ungar [12, 13, 14]. Einsteins addition in (3) of parallel relativistically admissible velocities is found in his 1905 paper that founded special relativity theory [1]. Also the general case of Einsteins velocity addition of relativistically admissible velocities which need not be parallel, (2), is found in Einsteins 1905 paper. Euclidean 3- vector algebra was not so widely known in 1905 and, consequently, was not used by Einstein. Einstein calculated in [2] the behavior of the velocity components parallel and orthogonal to the relative velocity between inertial systems, which is as close as one can get to the vectorial version (2) without vectors.

Most texts on special and general relativity present the Einstein velocity addition formula only for parallel velocities for simplicity (and usually under the assumption that the motion is in the direction of a coordinate axis). However, there are some outstanding exceptions to this, where the Einstein velocity addition law is presented for the general case when velocities need not be collinear. In its full generality, thus, the Einstein relativity groupoid is neither commutative nor associative.

The groupoid of classical velocities $(\mathbb{R}^3, +)$ under ordinary vector addition forms a commutative group, as opposed to Einsteins groupoid of relativistically admissible velocities $(\mathbb{R}_c^3, \oplus_E)$ which, under Einsteins velocity addition, does not form a group. Is the breakdown of associativity in the Einstein velocity addition associated with loss of mathematical regularity? To see that this is not the case, a missing link between vector addition and Einsteins addition was invoked, this is the Thomas precession.

We shall now take the definitions of some notions and concepts in quasi-group and loop theory. Let L be a non-empty set. Define a binary operation $(\cdot)$ on L . If $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid. If the equations:

$$a \cdot x = b \quad \text{and} \quad y \cdot a = b$$

have unique solutions for x and y respectively, then $(L, \cdot)$ is called a quasi-group. Now, if there exists a unique element $e \in L$ called the identity element such that for all $x \in L$, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop. For each $x \in L$, the elements $x^\rho = xJ_\rho \in L$ and $x^\lambda = xJ_\lambda \in L$ such that $xx^\rho = e$ and $x^\lambda x = e$ are called the right and left inverse elements of x respectively. In case $x^\lambda = x^\rho$ or $J_\rho = J_\lambda$ then, we simply write $x^\lambda = x^\rho = x^{-1}$ or $J_\rho = J_\lambda = J$ and refer to x^{-1} as the inverse of x and J as the inverse mapping.

A loop $(L, \cdot)$ is called a right Bol loop (RBL) if it satisfies the identity

$$(xy \cdot z)y = x(yz \cdot y). \quad (4)$$

A loop $(L, \cdot)$ is called a left Bol loop (LBL) if it satisfies the identity

$$y(z \cdot yx) = (y \cdot zy)x. \quad (5)$$

A loop $(L, \cdot)$ is called a Moufang loop (ML) if it satisfies the identity

$$(xy) \cdot (zx) = (x \cdot yz)x. \quad (6)$$

A loop $(L, \cdot)$ is called an extra loop (EL) if it satisfies the identity

$$(xy \cdot z)x = x(y \cdot zx) \quad (7)$$

A loop $(L, \cdot)$ is called an right central loop (RCL) if it satisfies the identity

$$yz \cdot xx = y(zx \cdot x) \quad (8)$$

A loop $(L, \cdot)$ is called an left central loop (LCL) if it satisfies the identity

$$(y \cdot xx)z = y(x \cdot xz) \quad (9)$$

A loop $(L, \cdot)$ is called an central loop (CL) if it satisfies the identity

$$(yx \cdot x)z = y(x \cdot xz) \quad (10)$$

A loop $(L, \cdot)$ is called an right nuclear square loop (RNSCL) if it satisfies the identity

$$yz \cdot xx = y(z \cdot xx) \quad (11)$$

A loop $(L, \cdot)$ is called an left nuclear square loop (LNSL) if it satisfies the identity

$$xx \cdot yz = (xx \cdot y)z \quad (12)$$

A loop $(L, \cdot)$ is called an middle nuclear square loop (MNSL) if it satisfies the identity

$$y(xx \cdot z) = (y \cdot xx)z \quad (13)$$

A loop $(L, \cdot)$ is called a left alternative property loop (LAPL) if it satisfies the identity

$$(xx \cdot yz) = x(x \cdot yz) \quad (14)$$

A loop $(L, \cdot)$ is called a right alternative property loop (RAPL) if it satisfies the identity

$$(yz \cdot xx) = (yz \cdot x)x \quad (15)$$

A loop $(L, \cdot)$ is called a flexible loop (FL) if it satisfies the identity

$$x(yz \cdot x) = (x \cdot yz)x \quad (16)$$

All the varieties of loops defined above fall into the class of loops called loops of Bol-Moufang type (Fenyves [3, 4], Phillips and Vojtěchovský [11]).

We now state the definitions of some inverse properties in loops. A loop $(L, \cdot)$ is called a right inverse property loop (RIPL) if it satisfies right inverse property (RIP)

$$(yx)x^{\rho} = y \quad (17)$$

A loop $(L, \cdot)$ is called a left inverse property loop (LIPL) if it satisfies left inverse property (LIP)

$$x^{\lambda}(xy) = y \quad (18)$$

A loop $(L, \cdot)$ is called a weak inverse property loop (WIPL) if it obeys the identity

$$x(yx)^\rho = y^\rho \quad \text{or} \quad (xy)^\lambda x = y^\lambda \quad (19)$$

A loop $(L, \cdot)$ is called a cross inverse property loop (CIPL) if it obeys the identity

$$xy \cdot x^\rho = y \text{ or } x \cdot yx^\rho = y \text{ or } x^\lambda \cdot (yx) = y \text{ or } x^\lambda y \cdot x = y \quad (20)$$

A loop $(L, \cdot)$ is called an automorphic inverse property loop (AIPL) if it satisfies automorphic inverse property (AIP)

$$(xy)^{-1} = x^{-1}y^{-1} \quad (21)$$

A loop $(L, \cdot)$ in which the mapping $x \mapsto x^2$ is a permutation, is called a Bruck loop if it is both a Bol loop and either AIPL or obeys the identity $xy^2 \cdot x = (yx)^2$.

The right nucleus of L is defined by $N_\rho(L, \cdot) = \{x \in L \mid zy \cdot x = z \cdot yx \ \forall y, z \in L\}$. The left nucleus of L is defined by $N_\lambda(L, \cdot) = \{a \in L : ax \cdot y = a \cdot xy \ \forall x, y \in L\}$. The middle nucleus of L is defined by $N_\mu(L, \cdot) = \{a \in L : ya \cdot x = y \cdot ax \ \forall x, y \in L\}$. The nucleus of L is defined by $N(L, \cdot) = N_\lambda(L, \cdot) \cap N_\rho(L, \cdot) \cap N_\mu(L, \cdot)$.

The left and right translation maps on a loop $(L, \cdot)$ are the bijections $L_x : L \rightarrow L$ and $R_x : L \rightarrow L$, respectively defined as $yR_x = y \cdot x = yx$ and $yL_x = x \cdot y = xy$ for all $x, y \in L$.

The set of all automorphisms of a loop $(L, \cdot)$ forms a group called the automorphism group, denoted by $AUM(L, \cdot)$.

Motivated by the definition of a group, the key features of Einsteins addition are abstracted and placed in the following formal definition of a gyrogroup.

Definition 1.1 (*Gyrogroup*)

A groupoid $(G, \oplus)$ is a gyrogroup if it obeys the following axioms:

Left Identity: there exists an element 0 satisfying $0 \oplus x$ for all $x \in G$;

Left Inverse: for each $x \in G$, there exists $\ominus x \in G$ such that $\ominus x \oplus x = 0$;
and

under the operation $gyr : G \times G \rightarrow AUM(G, \oplus)$ called the gyrooperation, every pair $x, y \in G$ generates the map $gyr[x, y] : G \rightarrow G$ called the Thomas gyration or gyroautomorphism which satisfies

Left Gyroassociative: $x \oplus (y \oplus z) = (x \oplus y) \oplus gyr[x, y](z)$ for all $x, y, z \in G$;

Gyroautomorphism: $gyr[x, y] \in AUM(G, \oplus)$ for all $x, y \in G$;

Left Loop Property: $gyr[x, y] = gyr[x \oplus y, y]$ for all $x, y \in G$.

In full analogy with groups, gyrogroups are classified into gyrocommutative and non-gyrocommutative gyrogroups. The definition of gyrocommutativity in gyrogroups follows.

Definition 1.2 (*Noncommutative Gyrogroup*)

A gyrogroup $(G, \oplus)$ is said to be gyrocommutative if

Gyrocommutative: $x \oplus y = gyr[x, y](y \oplus x)$ for all $x, y \in G$.

Example 1.1 (*The Einstein Gyrogroup*)

Consider the groupoid $(\mathbb{R}_c^3, \oplus_E)$ with $\oplus_E$ defined by (2). A generalized form of it is the groupoid $(\mathbb{V}_c, \oplus_E)$ where $\mathbb{V}_c$ is an open c -ball of a real inner product space $\mathbb{V}$ and called the the Einstein groupoid. It is a gyrocommutative gyrogroup.

It is the the first known example of a non-group gyrogroup that motivated the introduction of the gyrogroup notion A. A. Ungar.

Strictly speaking, among all loops which are related to gyrogroups and gyrocommutative gyrgroups, only the left Bol loop is of Bol-Moufang type. The aim of the present study is to investigate the relationship between gyrogroups, gyrocommutative gyrgroups and other loops of Bol-Moufang types such as RAPLs, FLs, WIPLs, RCLs, LCLs, CLs, RNSLs, LNSLs, MNSLs, RBLs, MLs and ELs. This is with the objective of characterizing gyrogroups' and gyrocommutative gyrgroups' Thomas Precession anew and unveiling the possible meaning of this in relativity theory. Some new properties of the Thomas Precession of some loops of Bol-Moufang type gyrogroup and gyrocommutative gyrgroup are also deduced.

2 Methodology

The theory of gyrogroups and gyrocommutative have been extensively studied by A. A. Ungar and majority of the results on it can be found in [15, 17, 18, 19, 16]. Some of them which are important for this present work are highlighted below. They shall be judiciously used.

Theorem 2.1 *Let $(G, \oplus)$ be a gyrogroup. Then the following are true.*

Identity: *there exists a unique element $0 \in G$ satisfying $0 \oplus x = x \oplus 0 = x$ for all $x \in G$.*

Inverse: *for each $x \in G$, there exists a unique $\ominus x \in G$ such that $\ominus x \oplus x = x \oplus (\ominus x) = 0$.*

$\text{gyr}[x, y]z = \ominus(x \oplus y) \oplus (x \oplus (y \oplus z))$ for all $x, y, z \in G$.

Right Gyroassociative: *$(x \oplus y) \oplus z = x \oplus (y \oplus \text{gyr}[y, x](z))$ for all $x, y, z \in G$.*

Right Loop Property: *$\text{gyr}[x, y] = \text{gyr}[x, y \oplus x]$ for all $x, y \in G$.*

$\text{gyr}^{-1}[x, y] = \text{gyr}[y, x]$ for all $x, y \in G$.

Theorem 2.2 *Let $(G, \oplus)$ be a gyrogroup. For any $x, y, z \in G$, the following are true.*

1. *$x \oplus y = x \oplus z$ implies $y = z$ and $y \oplus x = z \oplus x$ implies $y = z$.*
2. *$\text{gyr}[0, x] = \text{gyr}[\ominus x, x] = \text{gyr}[x, 0] = \text{gyr}[x, x] = I$.*
3. *LIP: $\ominus x \oplus (x \oplus y) = y$.*

Theorem 2.3 *Let $(G, \oplus)$ be a gyrogroup. For any $x, y, z \in G$ and $A \in \text{AUM}(G, \oplus)$, the following gyration identities are true.*

$$\text{gyr}[x, y \oplus z]\text{gyr}[y, z] = \text{gyr}[x \oplus y, \text{gyr}[x, y]z]\text{gyr}[x, y] \quad (22)$$

$$\text{gyr}[x, y]y = \ominus[\ominus(x \oplus y) \oplus x] \quad (23)$$

$$\text{Agyr}[x, y] = \text{gyr}[Ax, Ay]A \quad (24)$$

Theorem 2.4 *Let $(G, \oplus)$ be a gyrocommutative gyrogroup. For any $x, y, z \in G$, the following gyration identities are true.*

$$\text{gyr}[x, y \oplus z] \text{gyr}[y, z] = \text{gyr}[x, y] \text{gyr}[y \oplus x, z] \quad (25)$$

$$\text{gyr}[x, y]y = (x \oplus y) \ominus x \quad (26)$$

Gyrocommutative gyrogroups are closely related to various loops like A-loops, Bol-loops, Bruck loops, K-loops, and Kikkawas homogeneous loops (see Kikkawa [7], Issa [5], Rózga [9], Kinyon and Jones [8], Kruezer and Wefelscheid [10]). Gyrogroups are a type of left Bol loop. Gyrocommutative gyrogroups are equivalent to K-loops (Kiechle [6]) although defined differently. A K-loop is a Bruck loop. A Bruck loop in which no element is of order 2 is a K-loop.

Some algebraic properties of left Bol loops which we shall find useful in this work are stated in the theorem below.

Theorem 2.5 *Let $(L, \cdot)$ be LBL.*

1. $(L, \cdot)$ is a LIPL.
2. $x^\lambda = x^\rho$ for all $x \in L$.
3. $(L, \cdot)$ is a LAPL.
4. $N_\lambda(L, \cdot) = N_\mu(L, \cdot)$.

3 Main Results

Lemma 3.1 *Let $(G, \oplus)$ be a gyrogroup. The following are equivalent.*

1. $(G, \oplus)$ is a FL.
2. $(G, \oplus)$ is a RAPL.
3. $(G, \oplus)$ is a WIPL.
4. $\text{gyr}[x, y]x = x$ for all $x, y \in G$.

5. $gyr[y, x]x = x$ for all $x, y \in G$.

Proof

We shall keep Theorem 2.1 and Theorem 2.2 in mind.

1. $\Leftrightarrow$ 4. $\Leftrightarrow$ 5.: $(G, \oplus)$ is a FL if and only if $x \oplus (y \oplus x) = (x \oplus y) \oplus x \Leftrightarrow (x \oplus y) \oplus gyr[x, y]x = (x \oplus y) \oplus x \Leftrightarrow gyr[x, y]x = x \Leftrightarrow gyr[y, x]x = x$.
2. $\Leftrightarrow$ 4. $\Leftrightarrow$ 5.: $(G, \oplus)$ is a RAPL if and only if $y \oplus (x \oplus x) = (y \oplus x) \oplus x \Leftrightarrow (y \oplus x) \oplus gyr[y, x]x = (y \oplus x) \oplus x \Leftrightarrow gyr[y, x]x = x \Leftrightarrow gyr[y, x]x = x$.
3. $\Leftrightarrow$ 5.: By (23), $gyr[x, y]y = \ominus[\ominus(x \oplus y) \oplus x] = y \Rightarrow \ominus(x \oplus y) \oplus x = \ominus y \Leftrightarrow (G, \oplus)$ is a WIPL.

Lemma 3.2 *Let $(G, \oplus)$ be a gyrogroup. The following are equivalent.*

1. $(G, \oplus)$ is a RNSL.
2. $gyr[z, y](x \oplus x) = x \oplus x$ for all $x, y, z \in G$.

Proof

We shall keep Theorem 2.1 and Theorem 2.2 in mind. $(G, \oplus)$ is a RNSL if and only if $(y \oplus z) \oplus (x \oplus x) = y \oplus (z \oplus (x \oplus x)) \Leftrightarrow (y \oplus z) \oplus (x \oplus x) = (y \oplus z) \oplus gyr[y, z](x \oplus x) \Leftrightarrow gyr[z, y](x \oplus x) = x \oplus x$.

Theorem 3.1 *Let $(G, \oplus)$ be a gyrogroup. The following are equivalent.*

1. $(G, \oplus)$ is an LCL.
2. $gyr[x \oplus x, y] = I$ for all $x, y \in G$.
3. $gyr[y, x \oplus x] = I$ for all $x, y \in G$.
4. $x \oplus x \in N_\lambda(L, \cdot)$ for all $x \in G$.
5. $x \oplus x \in N_\mu(L, \cdot)$ for all $x \in G$.

Proof

We shall keep Theorem 2.1 and Theorem 2.2 in mind. $(G, \oplus)$ is an LCL if and only if (9) is true if and only if

$$\begin{aligned} [y \oplus (x \oplus x)] \oplus z &= y \oplus [x \oplus (x \oplus z)] \Leftrightarrow \\ y \oplus ((x \oplus x) \oplus gyr[x \oplus x, y]z) &= y \oplus ((x \oplus x) \oplus gyr[x, x]z) \Leftrightarrow \\ gyr[x \oplus x, y]z &= z \Leftrightarrow gyr[x \oplus x, y] = I \end{aligned}$$

We need to use Theorem 2.5. Note that $gyr[x, y] = L_x L_y L_{y \oplus x}^{-1}$. So, $gyr[x \oplus x, y] = I \Leftrightarrow L_{x \oplus x} L_y = L_{y \oplus (x \oplus x)} \Leftrightarrow y \oplus [(x \oplus x) \oplus z] = [y \oplus (x \oplus x)] \oplus z$ for all $x, y, z \in G \Leftrightarrow (x \oplus x) \in N_\mu(G, \oplus) \Leftrightarrow (x \oplus x) \in N_\lambda(G, \oplus)$.

The proofs of 4. and 5. is the same as the proofs of 4. and 5. of Theorem 3.1.

Theorem 3.2 *Let $(G, \oplus)$ be a gyrogroup. The following are equivalent.*

1. $(G, \oplus)$ is an RCL.
2. $gyr[z, y](x \oplus x) = \ominus z \oplus [(z \oplus x) \oplus x]$ for all $x, y, z \in G$.
3. $x \oplus gyr[x, y \oplus z]x = gyr[y, z]x \oplus gyr[y, z]gyr[x, z]x$ for all $x, y, z \in G$.
4. $gyr[y, z](x \oplus x) = x \oplus gyr[x, y \oplus z]x$ for all $x, y, z \in G$.
5. $gyr[x, y]L_x gyr[x, y] = L_x$ for all $x, y, z \in G$.

Proof

We shall keep Theorem 2.1 and Theorem 2.2 in mind.

1. $\Leftrightarrow$ 2. $(G, \oplus)$ is an RCL if and only if $(y \oplus z) \oplus (x \oplus x) = y \oplus [(z \oplus x) \oplus x]$

$$\begin{aligned} \Leftrightarrow y \oplus [z \oplus gyr[z, y](x \oplus x)] &= y \oplus [(z \oplus x) \oplus x] \Leftrightarrow \\ z \oplus gyr[z, y](x \oplus x) &= (z \oplus x) \oplus x \Leftrightarrow gyr[z, y](x \oplus x) = \\ &\ominus z \oplus [(z \oplus x) \oplus x]. \end{aligned}$$

1. $\Leftrightarrow$ 3. $(G, \oplus)$ is an RCL if and only if $[(y \oplus z) \oplus x] \oplus x = y \oplus [(z \oplus x) \oplus x]$

$$\begin{aligned} &\Leftrightarrow (y \oplus z) \oplus [x \oplus \text{gyr}[x, y \oplus z]x] = y \oplus [(z \oplus x) \oplus x] \Leftrightarrow \\ &z \oplus [\text{gyr}[z, y]x \oplus \text{gyr}[z, y]\text{gyr}[x, y \oplus z]x] = z \oplus (x \oplus \text{gyr}[x, z]x) \Leftrightarrow \\ &x \oplus \text{gyr}[x, y \oplus z]x = \text{gyr}[y, z]x \oplus \text{gyr}[y, z]\text{gyr}[x, z]x. \end{aligned}$$

1. $\Leftrightarrow$ 4. $(G, \oplus)$ is an RCL if and only if $[(y \oplus z) \oplus x] \oplus x = y \oplus [z \oplus (x \oplus x)]$

$$\begin{aligned} &\Leftrightarrow (y \oplus z) \oplus [x \oplus \text{gyr}[x, y \oplus z]x] = (y \oplus z) \oplus \text{gyr}[y, z](x \oplus x) \Leftrightarrow \\ &\text{gyr}[y, z](x \oplus x) = x \oplus \text{gyr}[x, y \oplus z]x. \end{aligned}$$

1. $\Leftrightarrow$ 5. $(G, \oplus)$ is an RCL if and only if $[(y \oplus x) \oplus x] \oplus z = y \oplus [(x \oplus x) \oplus z]$

$$\begin{aligned} &\Leftrightarrow (y \oplus x) \oplus [x \oplus \text{gyr}[x, y \oplus x]z] = y \oplus [x \oplus (x \oplus z)] \Leftrightarrow \\ &y \oplus \left\{ x \oplus \text{gyr}[x, y](x \oplus \text{gyr}[x, y \oplus x]z) \right\} = y \oplus [x \oplus (x \oplus z)] \Leftrightarrow \\ &\text{gyr}[x, y](x \oplus \text{gyr}[x, y \oplus x]z) = (x \oplus z) \Leftrightarrow \text{gyr}[x, y]L_x \text{gyr}[x, y] = L_x. \end{aligned}$$

Remark 3.1 A gyrogroup is an RCL if and only if the conjugate of some gyrations by some left translation is a gyration. A gyrogroup is an RCL if and only if every gyration is a cross inverse element for some left translation.

Theorem 3.3 Let $(G, \oplus)$ be a gyrogroup. If $(G, \oplus)$ is an RCL, then

1. $\text{gyr}[z, y](x \oplus x) = x \oplus x$ for all $x, y, z \in G$. Hence, $(G, \oplus)$ is an RNSL.
2. $\text{gyr}[x, y]x = x$ for all $x, y \in G$. Hence, $(G, \oplus)$ is an FL and RAPL.
3. $(G, \oplus)$ is a WIPL.
4. $\text{gyr}[x \oplus x, x] = I$ for all $x \in G$.

Proof

1. In an RCL $(G, \oplus)$, $x \oplus x \in N_\rho(G, \oplus)$. So, by 2. of Theorem 3.2, $gyr[z, y](x \oplus x) = \ominus z \oplus [z \oplus (x \oplus x)] = x \oplus x$. Following Lemma 3.2, $(G, \oplus)$ is an RNSL.
2. Put $z = 0$ in 4. of Theorem 3.2 to get $gyr[y, 0](x \oplus x) = x \oplus x = x \oplus gyr[x, y \oplus 0]x = x \oplus gyr[x, y]x$ which implies $gyr[x, y]x = x$. The conclusion follows from Lemma 3.1.
3. By (23), $gyr[x, y]y = \ominus[\ominus(x \oplus y) \oplus x] = y \Rightarrow \ominus(x \oplus y) \oplus x = \ominus y$. So, $(G, \oplus)$ is a WIPL.
4. Using 5. of Theorem 3.2, $gyr[x, y]L_x gyr[x, y \oplus x] = L_x$. Substitute $y = x$ to get $gyr[x \oplus x, x] = I$.

Corollary 3.1 *A gyrocommutative gyrogroup RCL is a CIPL.*

Proof

Going by (26) and 2., $gyr[x, y]y = (x \oplus y) \ominus x = y$. Hence, the CIP holds.

Lemma 3.3 *Let $(G, \oplus)$ be a gyrogroup that is an RCL. If $(G, \oplus)$ is left distributive then, $gyr[x \oplus (y \oplus x), x \oplus x] = gyr[x, y] = gyr[(x \oplus y) \oplus (x \oplus x), x \oplus x]$ for all $x, y \in G$.*

Proof

$(G, \oplus)$ is left distributive if $L_x \in AUM(G, \oplus)$. So by (24), $L_x gyr[x, y \oplus x] = gyr[x \oplus x, x \oplus (y \oplus x)]L_x = gyr[x \oplus x, (x \oplus y) \oplus (x \oplus x)]L_x = gyr[y, x]L_x$. Hence, $gyr[x \oplus (y \oplus x), x \oplus x] = gyr[x, y] = gyr[(x \oplus y) \oplus (x \oplus x), x \oplus x]$.

Corollary 3.2 *A left distributive RCL gyrogroup of exponent 2 is a group.*

Proof

Following Lemma 3.3, $gyr[x \oplus (y \oplus x), x \oplus x] = gyr[x \oplus (y \oplus x), 0] = I = gyr[x, y]$.

Theorem 3.4 *Let $(G, \oplus)$ be a gyrogroup. $(G, \oplus)$ is a CL if and only if $gyr[x, y]L_x gyr[x, y] = L_x$ for all $x, y, z \in G$.*

Proof

We shall keep Theorem 2.1 and Theorem 2.2 in mind. $(G, \oplus)$ is a CL if and only if $[(y \oplus x) \oplus x] \oplus z = y \oplus [x \oplus (x \oplus z)]$

$$\begin{aligned} &\Leftrightarrow (y \oplus x) \oplus [x \oplus gyr[x, y \oplus x]z] = (y \oplus x) \oplus gyr[y, x](x \oplus z) \Leftrightarrow \\ &L_x gyr[x, y \oplus x] = gyr[y, x]L_x \Leftrightarrow L_x gyr[x, y] = gyr[y, x]L_x \Leftrightarrow \\ &gyr[x, y]L_x gyr[x, y] = L_x. \end{aligned}$$

Theorem 3.5 *Let $(G, \oplus)$ be a gyrogroup. The following are equivalent.*

1. $(G, \oplus)$ is an EL.
2. $z \oplus x = gyr[y, x]z \oplus gyr[y, x]gyr[z, x \oplus y]x$ for all $x, y, z \in G$.
3. $gyr[x, y]L_x gyr[x, y] = L_x$ for all $x, y, z \in G$.
4. $z \oplus x = gyr[y, x]z \oplus gyr[y, x]gyr[z, x]gyr[x \oplus z, y]x$ for all $x, y, z \in G$.

Proof

We shall keep Theorem 2.1 and Theorem 2.2 in mind.

1. $\Leftrightarrow$ 2. $(G, \oplus)$ is an EL if and only if $[(x \oplus y) \oplus z] \oplus x = x \oplus [y \oplus (z \oplus x)]$

$$\begin{aligned} &\Leftrightarrow (x \oplus y) \oplus [z \oplus gyr[z, x \oplus y]x] = (x \oplus y) \oplus gyr[x, y](z \oplus x) \Leftrightarrow \\ &z \oplus gyr[z, x \oplus y]x = gyr[x, y](z \oplus x) \Leftrightarrow \\ &gyr[y, x]z \oplus gyr[y, x]gyr[z, x \oplus y]x = z \oplus x. \end{aligned}$$

1. $\Leftrightarrow$ 3. $(G, \oplus)$ is an EL if and only if $(x \oplus y) \oplus (x \oplus z) = x \oplus [(y \oplus x) \oplus z]$

$$\begin{aligned} &\Leftrightarrow x \oplus (y \oplus gyr[y, x](x \oplus z)) = x \oplus [y \oplus (x \oplus gyr[x, y]z)] \Leftrightarrow \\ &gyr[y, x](x \oplus z) = x \oplus gyr[x, y]z \Leftrightarrow gyr[y, x]L_x = L_x gyr[x, y] \Leftrightarrow \\ &gyr[x, y]L_x gyr[x, y] = L_x. \end{aligned}$$

1. $\Leftrightarrow$ 4. $(G, \oplus)$ is an EL if and only if $(y \oplus x) \oplus (z \oplus x) = [y \oplus (x \oplus z)] \oplus x$

$$\begin{aligned} &\Leftrightarrow y \oplus [x \oplus gyr[x, y](z \oplus x)] = y \oplus [(x \oplus z) \oplus gyr[x \oplus z, y]x] \Leftrightarrow \\ &x \oplus gyr[x, y](z \oplus x) = (x \oplus z) \oplus gyr[x \oplus z, y]x \Leftrightarrow \\ &gyr[x, y](z \oplus x) = z \oplus gyr[z, x]gyr[x \oplus z, y]x \Leftrightarrow \\ &z \oplus x = gyr[y, x]z \oplus gyr[y, x]gyr[z, x]gyr[x \oplus z, y]x. \end{aligned}$$

Theorem 3.6 *Let $(G, \oplus)$ be a gyrocommutative gyrogroup. $(G, \oplus)$ is a EL if and only if $z \oplus x = \text{gyr}[y, x]z \oplus \text{gyr}[y, x]\text{gyr}[z, x \oplus y]\text{gyr}[x, y]x$ for all $x, y, z \in G$.*

Proof

Going by 4. of Theorem 3.5 and (25),

$$z \oplus x = \text{gyr}[y, x]z \oplus \text{gyr}[y, x]\text{gyr}[z, x]\text{gyr}[x \oplus z, y]x \Leftrightarrow z \oplus x = \text{gyr}[y, x]z \oplus \text{gyr}[y, x]\text{gyr}[z, x \oplus y]\text{gyr}[x, y]x.$$

Theorem 3.7 *Let $(G, \oplus)$ be a gyrogroup. The following are equivalent.*

1. $(G, \oplus)$ is a LNSL.
2. $(G, \oplus)$ is a MNSL.
3. $\text{gyr}[x \oplus x, y] = I$ for all $x, y \in G$.
4. $\text{gyr}[y, x \oplus x] = I$ for all $x, y \in G$.
5. $x \oplus x \in N_\lambda(L, \cdot)$ for all $x \in G$.
6. $x \oplus x \in N_\mu(L, \cdot)$ for all $x \in G$.

Proof

We shall keep Theorem 2.1 and Theorem 2.2 in mind.

1. $\Leftrightarrow$ 3. $\Leftrightarrow$ 4. $(G, \oplus)$ is an LNSL if and only if $(x \oplus x) \oplus (y \oplus z) = [(x \oplus x) \oplus y] \oplus z$

$$\Leftrightarrow (x \oplus x) \oplus (y \oplus z) = (x \oplus x) \oplus (y \oplus \text{gyr}[y, x \oplus x]z) \Leftrightarrow \text{gyr}[y, x \oplus x] = I \Leftrightarrow \text{gyr}[x \oplus x, y] = I.$$

2. $\Leftrightarrow$ 3. $\Leftrightarrow$ 4. $(G, \oplus)$ is an MNSL if and only if $(y \oplus (x \oplus x)) \oplus z = y \oplus [(x \oplus x) \oplus z]$

$$\Leftrightarrow (y \oplus (x \oplus x)) \oplus z = (y \oplus (x \oplus x)) \oplus \text{gyr}[y, x \oplus x]z \Leftrightarrow \text{gyr}[y, x \oplus x] = I \Leftrightarrow \text{gyr}[x \oplus x, y] = I.$$

Corollary 3.3 *Let $(G, \oplus)$ be a gyrogroup. The following are equivalent.*

1. $(G, \oplus)$ is a LCL.
2. $(G, \oplus)$ is a LNSL.
3. $(G, \oplus)$ is a MNSL.
4. $\text{gyr}[x \oplus x, y] = I$ for all $x, y \in G$.
5. $\text{gyr}[y, x \oplus x] = I$ for all $x, y \in G$.
6. $x \oplus x \in N_\lambda(L, \cdot)$ for all $x \in G$.
7. $x \oplus x \in N_\mu(L, \cdot)$ for all $x \in G$.

Proof

This is achieved by Theorem 3.1 and Theorem 3.7.

Corollary 3.4 *Let $(G, \oplus)$ be a gyrogroup. The following are equivalent.*

1. $(G, \oplus)$ is an EL.
2. $(G, \oplus)$ is a CL.
3. $(G, \oplus)$ is a RCL.
4. $\text{gyr}[z, y](x \oplus x) = \ominus z \oplus [(z \oplus x) \oplus x]$ for all $x, y, z \in G$.
5. $x \oplus \text{gyr}[x, y \oplus z]x = \text{gyr}[y, z]x \oplus \text{gyr}[y, z]\text{gyr}[x, z]x$ for all $x, y, z \in G$.
6. $\text{gyr}[y, z](x \oplus x) = x \oplus \text{gyr}[x, y \oplus z]x$ for all $x, y, z \in G$.
7. $\text{gyr}[x, y]L_x\text{gyr}[x, y] = L_x$ for all $x, y, z \in G$.
8. $z \oplus x = \text{gyr}[y, x]z \oplus \text{gyr}[y, x]\text{gyr}[z, x \oplus y]x$ for all $x, y, z \in G$.
9. $z \oplus x = \text{gyr}[y, x]z \oplus \text{gyr}[y, x]\text{gyr}[z, x]\text{gyr}[x \oplus z, y]x$ for all $x, y, z \in G$.

Proof

This is achieved by Theorem 3.2, Theorem 3.4 and Theorem 3.5.

Theorem 3.8 *Let $(G, \oplus)$ be a gyrogroup. $(G, \oplus)$ is an RBL if and only if $(x \oplus z) \oplus \text{gyr}[x \oplus z, y]x = (x \oplus \text{gyr}[y, x]z) \oplus x$ for all $x, y, z \in G$.*

Proof

We shall keep Theorem 2.1 and Theorem 2.2 in mind. $(G, \oplus)$ is an RBL if and only if

$$\begin{aligned} [(y \oplus x) \oplus z] \oplus x &= y \oplus ((x \oplus z) \oplus x) \Leftrightarrow \\ [y \oplus (x \oplus \text{gyr}[x, y]z)] \oplus x &= y \oplus ((x \oplus z) \oplus x) \Leftrightarrow \\ y \oplus \{ (x \oplus \text{gyr}[x, y]z) \oplus \text{gyr}[x \oplus \text{gyr}[x, y]z, y]x \} &= y \oplus ((x \oplus z) \oplus x) \Leftrightarrow \\ (x \oplus \text{gyr}[x, y]z) \oplus \text{gyr}[x \oplus \text{gyr}[x, y]z, y]x &= (x \oplus z) \oplus x \Leftrightarrow \\ (x \oplus z) \oplus \text{gyr}[x \oplus z, y]x &= (x \oplus \text{gyr}[y, x]z) \oplus x. \end{aligned}$$

Theorem 3.9 *Let $(G, \oplus)$ be a gyrogroup. The following are equivalent.*

1. $(G, \oplus)$ is an ML.
2. $(G, \oplus)$ is an RBL.
3. $y \oplus \text{gyr}[y, x](z \oplus x) = (y \oplus z) \oplus \text{gyr}[y \oplus z, x]x$ for all $x, y, z \in G$.
4. $\text{gyr}[y, x]z \ominus [\ominus(y \oplus x) \oplus y] = \ominus y \oplus \left[(y \oplus z) \ominus [\ominus((y \oplus z) \oplus x) \oplus (y \oplus z)] \right]$ for all $x, y, z \in G$.
5. $z \oplus x = \text{gyr}[x, y]z \oplus \text{gyr}[x, y]\text{gyr}[z, x]x$ for all $x, y, z \in G$.
6. $\text{gyr}[x, y] = L_x \text{gyr}[x, x \oplus y] L_x^{-1}$ for all $x, y, z \in G$.
7. $z \oplus x = \text{gyr}[x, y]z \oplus \text{gyr}[x, y]\text{gyr}[z, y \oplus x]x$ for all $x, y, z \in G$.
8. $(x \oplus z) \oplus \text{gyr}[x \oplus z, y]x = (x \oplus \text{gyr}[y, x]z) \oplus x$ for all $x, y, z \in G$.

Proof

We shall keep Theorem 2.1 and Theorem 2.2 in mind.

1. $\Leftrightarrow$ 2. $(G, \oplus)$ is a ML if and only if $(G, \oplus)$ is a RBL and a LBL if and only if $(G, \oplus)$ is a RBL.

$$\begin{aligned}
 1. \Leftrightarrow 3. \quad (G, \oplus) \text{ is an ML if and only if } (x \oplus y) \oplus (z \oplus x) &= (x \oplus (y \oplus z)) \oplus x \\
 \Leftrightarrow x \oplus [y \oplus \text{gyr}[y, x](z \oplus x)] &= x \oplus [(y \oplus z) \oplus \text{gyr}[y \oplus z, x]x] \Leftrightarrow \\
 y \oplus \text{gyr}[y, x](z \oplus x) &= (y \oplus z) \oplus \text{gyr}[y \oplus z, x]x.
 \end{aligned}$$

1. $\Leftrightarrow$ 4. Applying (23) to 3., we get 4.

$$\begin{aligned}
 1. \Leftrightarrow 5. \quad (G, \oplus) \text{ is an ML if and only if } (x \oplus y) \oplus (z \oplus x) &= x \oplus [(y \oplus z) \oplus x] \\
 \Leftrightarrow x \oplus [y \oplus \text{gyr}[y, x](z \oplus x)] &= x \oplus [y \oplus (z \oplus \text{gyr}[z, x]x)] \Leftrightarrow \\
 z \oplus x &= \text{gyr}[x, y]z \oplus \text{gyr}[x, y]\text{gyr}[z, x]x.
 \end{aligned}$$

$$\begin{aligned}
 1. \Leftrightarrow 6. \quad (G, \oplus) \text{ is an ML if and only if } [(x \oplus y) \oplus x] \oplus z &= x \oplus [y \oplus (x \oplus z)] \\
 \Leftrightarrow (x \oplus y) \oplus (x \oplus \text{gyr}[x, x \oplus y]z) &= (x \oplus y) \oplus \text{gyr}[x, y](x \oplus z) \Leftrightarrow \\
 x \oplus \text{gyr}[x, x \oplus y]z &= \text{gyr}[x, y](x \oplus z) \Leftrightarrow \text{gyr}[x, y] = L_x \text{gyr}[x, x \oplus y]L_x^{-1}.
 \end{aligned}$$

$$\begin{aligned}
 1. \Leftrightarrow 7. \quad (G, \oplus) \text{ is an ML if and only if } [(y \oplus x) \oplus z] \oplus x &= y \oplus (x \oplus (z \oplus x)) \\
 \Leftrightarrow (y \oplus x) \oplus [z \oplus \text{gyr}[z, y \oplus x]x] &= (y \oplus x) \oplus \text{gyr}[y, x](z \oplus x) \Leftrightarrow \\
 z \oplus \text{gyr}[z, y \oplus x]x &= (y \oplus x) \oplus \text{gyr}[y, x](z \oplus x) \Leftrightarrow \\
 z \oplus x &= \text{gyr}[x, y]z \oplus \text{gyr}[x, y]\text{gyr}[z, y \oplus x]x
 \end{aligned}$$

1. $\Leftrightarrow$ 8. This follows by Theorem 3.8 and 2..

Remark 3.2 *A gyrogroup is an ML if and only if the conjugate of some gyrations by some left translation is a gyration.*

Theorem 3.10 *Let $(G, \oplus)$ be a gyrocommutative gyrogroup. $(G, \oplus)$ is an ML if and only if $\text{gyr}[y, x]z \ominus [(y \oplus x) \ominus y] = \ominus y \oplus [(y \oplus z) \oplus [((y \oplus z) \oplus x) \oplus (y \oplus z)]]$ for all $x, y, z \in G$.*

Proof

Apply (26) to 3. of Theorem 3.9.

Lemma 3.4 *Let $(G, \oplus)$ be a gyrogroup that is an ML. If $(G, \oplus)$ is left distributive then, $gyr[x \oplus x, x \oplus (x \oplus y)] = gyr[x, y] = gyr[x \oplus x, (x \oplus x) \oplus (x \oplus y)]$ for all $x, y \in G$.*

Proof

$(G, \oplus)$ is left distributive if $L_x \in AUM(G, \oplus)$. So by (24), $L_x gyr[x, x \oplus y] = gyr[x \oplus x, x \oplus (x \oplus y)]L_x = gyr[x \oplus x, (x \oplus x) \oplus (x \oplus y)]L_x = gyr[x, y]L_x$. Hence, $gyr[x \oplus x, x \oplus (x \oplus y)] = gyr[x, y] = gyr[x \oplus x, (x \oplus x) \oplus (x \oplus y)]$.

Corollary 3.5 *A left distributive ML gyrogroup of exponent 2 is a group.*

Proof Following Lemma 3.4, $gyr[x \oplus x, x \oplus (x \oplus y)] = gyr[0, x \oplus (x \oplus y)] = I = gyr[x, y]$.

Corollary 3.6 *Let $(G, \oplus)$ be a gyrogroup. $(G, \oplus)$ is a central loop if and only if anyone of the equivalent conditions in Corollary 3.4 and anyone of the equivalent conditions in Corollary 3.3 are true.*

Proof

This is based on the fact that $(G, \oplus)$ is a CL if and only if $(G, \oplus)$ is a RCL and $(G, \oplus)$ is a LCL.

Corollary 3.7 *Let $(G, \oplus)$ be a gyrogroup. $(G, \oplus)$ is a extra loops if and only if anyone of the equivalent conditions in Corollary 3.4 and anyone of the equivalent conditions in Theorem 3.9 are true.*

Proof

This is based on the fact that $(G, \oplus)$ is a EL if and only if $(G, \oplus)$ is a ML and a CL.

The results in this section are summarised in Table 1.

Gyroggroups	
Equivalent Loop Properties	Equivalent Gyration Property
RAPL,WIPL,FL	$gyr[x, y]x = x$
RNSL	$gyr[z, y](x \oplus x) = x \oplus x$
LCL,LNSL,MNSL	$gyr[x \oplus x, y] = I$
EL*,CL*,RCL*	$gyr[x, y]L_xgyr[x, x \oplus y] = L_x$
RBL,ML	$gyr[x, y] = L_xgyr[x, x \oplus y]L_x^{-1}$

Table 1: Equivalent Loop Properties and Gyration for Gyroggroups

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